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RADC-TR-66-522

POSITION FIXING BY TIME OF ARRIVAL

Harley R. Smith

TECHNICAL REPORT NO. RADC-TR-66-522
November 1966

Rome Air Development Center
Research and Technology Division
Air Force Systems Command
Griffiss Air Force Base, New York

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UNCLASSIFIED

RADC-TR-66-522 November 1966

Errata - December 1966

The following information is applicable to RADC-TR-66-522, entitled, "Position Fixing by Time of Arrival", unclassified report, dated November 1966.

Page 17, 18 - Third paragraph should read: The QCEP, for the case α =50%--------------

Page 27 - At the end of the second paragraph add Figure 5f i.e., "--- QCEP equivalency case, Figure 5(f)".

Pages 33, 34 and 42 (Figures 5f and 6f) - The ordinate (vertical) scale should have been numbered between σ_s and 10 σ_s the same as the abscissa scale was between 1° and 10°.

Page 49 - The itemized assumptions which involve changes should read as follows:

Item 2, $\Delta\theta$ = 130° (Fig 4)
Item 3, $\Delta\phi$ = 25° (Fig 4)
Item 4, $\Delta\xi$ = 25° (Fig 4)

Pages 66 and 67, 68; Figures 12 and 13 - The adscissa scales which were omitted should be logarithmically proportioned between β = 1 and β = 10.

Page 73,74 - In the footnote the word "mode" should have read "model".

Page 85,86 - The equation for θ_y should have read:

$$\theta_y = \theta_x + \frac{\Delta\theta_m}{2} + \frac{\Delta\theta_n}{2} = 0.5° + 0.5° + 0.5°$$

Page 87 - Equation (A-5) and Figure A-1, the γ should have been a lower case gamma γ.

Page 109 - Equation D-14 is missing a square bracket,], between the curled brace } and the word exp.

Equation D-16 - A curled brace, {, is missing between the integral sign and the symbol ϕ.

Page 156 - The denominator of the second equation should have appeared as -- $b^2 \chi F_{MAX}$

Page 169 - The expression between K-5 and K-6 is an exaggeration which is not necessary to the development.

Page 172 - K-29 has subscripts in error; should read:

$$\frac{d}{a} = \frac{\theta_y}{\theta_z} = \frac{\theta_y}{\theta_x - \frac{\Delta\theta_m}{2} - \frac{\Delta\theta_n}{2}}$$

POSITION FIXING BY TIME OF ARRIVAL

Harley R. Smith

AIR FORCE, Cato Show Prtg. Co. - 12/8/66 - 141

FOREWORD

This technical report was prepared under Rome Air Development Center Project 4662.

The author wishes to acknowledge the interest and helpful suggestions provided by Alfred S. Kobos and Donald E. Savage. Also, the computer analysis work of David F. Trad was a significant contribution.

RADC Project Engineer was Harley R. Smith (EMCA).

Distribution of this report to the general public is withheld because it can significantly diminish the technological lead time of the United States and friendly foreign nations by revealing formulas and techniques having a potential strategic value not generally known through the world.

This report has been reviewed and is approved.

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ABSTRACT

An engineering model is developed of the error contours encountered in position fixing using synchronous time of arrival data. Normal distribution is emphasized. Probabilities of fixes occurring within circles and ellipses are determined using the natural oblique coordinates associated with the measuring system. Comparisons are made of three-observer and four-observer configurations. Examples are provided and applications are discussed.

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I. INTRODUCTION

In general, any event generating a signal which is sharply defined in time, and which is propagated with known velocity to three or more fixed observers, has a spatial position which can be calculated. Conversely, from the reception at a single point (observer) of three or more sharply defined time signals emanating from synchronized fixed stations, a spatial position can be determined (i.e., LORAN).

The determination of the position fix is accomplished by communicating the synchronous time of arrival (TOA) data to some common point of intelligence where the lines of position and their intersection can be calculated.

The accuracy of calculations is, in general, directly proportional to the sharpness of the time signature and the separation (in space) of the observers or the stations.

There have been numerous treatments (references 1, 2, 3, 4, 5, 6, and 7) of the physical systems which provide measurements for calculating a position fix. Most of these also discuss the mathematics of calculation of the fix, as well as the various error factors encountered and their causes. Analyses of the impact of errors on the spatial fix geometry has, however, for the most part been treated as incidental to these other objectives.

Therefore, this effort was undertaken in response to an apparent need for the derivation of a model of the geometric analysis of errors. The prime purpose was to make this model thorough enough to explain the various factors and parameters and provide derivations of all functions and quantities necessary for understanding and using such a model. In the following description of the method of attack, most of the analyses and results obtained did not appear to be available today, as indicated by research of the references and many other similar documents.

Engineering simplifications are used for greater insight, provided the end results are not contaminated by more than 5% error. Following a review of the fundamentals, a new tool of error geometry resulting from a uniform or constant error density is furnished as a means of making comparisons with other density functions. Calculations and curves are provided for the Quasi-Circular Error Probability (QCEP) radius for all values of probability in addition to the usual probable error of $\alpha = 50\%$. Determination is made of the Elliptical Error Probability (EEP) ellipse in terms of the probability of error, ratio of (space to time) gradients of the base lines, and the standard deviation of spatial displacement per base line. A comparison is then made of QCEP's and EEP's.

Since both the QCEP and EEP analysis is performed for dependent (three observers) and independent (four observers) lines of position, a comparison shows that, contrary to intuition, the three observers emerge superior to four.

Derivation and application of elliptical transforms from a rectangular to an oblique coordinate system is provided. Probabilistic/geometrical aspects of interpreting single data sets (versus the usual distribution of large samples) are discussed. There is, in effect, a correction factor, which is derived, for the true divergence of the oblique coordinate system about the fix point. Examples are also given of applying this model including a possible deep space application.

II. FUNDAMENTALS

The most fundamental tool involved is the locus of points of a given time difference, i.e., difference in distance from a fixed pair of observers. This locus, as shown in Figure 1, is a hyperbola.

The mathematics involved in this vital system component are quite simple. If the propagation from point P to observers A and B is in a straight line at a velocity V, then, for a given hyperbola,

$$D_A - D_B = v(t_A - t_B) = \text{constant} \tag{1}$$

where t_A and t_B are the respective times of arrival (TOA's) measured to the same (synchronous) reference, and the distances D_A and D_B are also straight-line measurements.

It is important to note, for the sake of generalization, that we are not restricted to any plane of action. Drawing the two hyperbolas in the plane of the paper was purely arbitrary. The total loci of points are the surfaces of revolution obtained by revolving the hyperbolas, as shown in Figure 2, around the line A-B as an axis. If we wish to confine our interest to any particular surface, which supposedly contains the position fix, we may limit the loci to those of the lines of intersection of the particular surface and of the hyperboloid of revolution. The major part of this report is limited to a plane representation of the earth, or space, as such a particular surface.

If the paths of propagation are not straight lines, but have known radii of curvature, the loci surfaces may still be defined and will, in general, be hyperbolic in nature. For example, in the case of an assumed sphere such as the earth, the radius of curvature is the radius of the earth and the propagation paths are (assumed) great circles. The hyperbolic surfaces will, accordingly, have to be modified. In effect, they would be generated from a plane hyperbolic figure whose distances D_A and D_B represent great circle arc distances instead of plane straight line distances.

While the development of this concept might be of considerable future benefit, it is not an objective of this report. Suffice it to say that the change in the difference of path lengths is not linear and is given by

$$\Delta(D_A - D_B) = R(\theta_1 - \theta_2) - 2R(\sin\theta_2/2 - \sin\theta_1/2) \tag{2}$$

where

R = radius of the earth

θ_1 and θ_2 represent the great circle distances of D_A and D_B

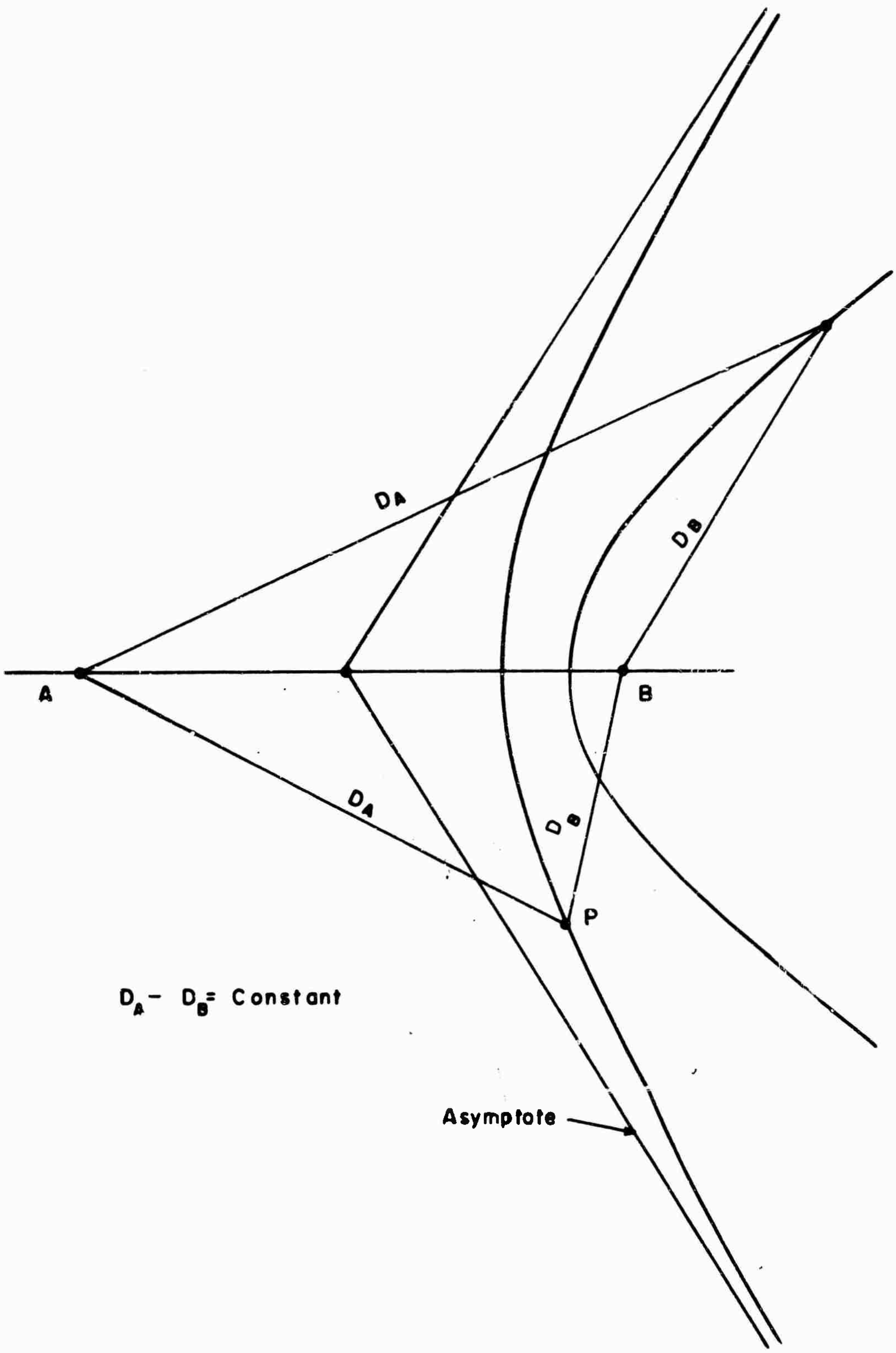

Figure 1. Fundamental Hyperbolic Geometry

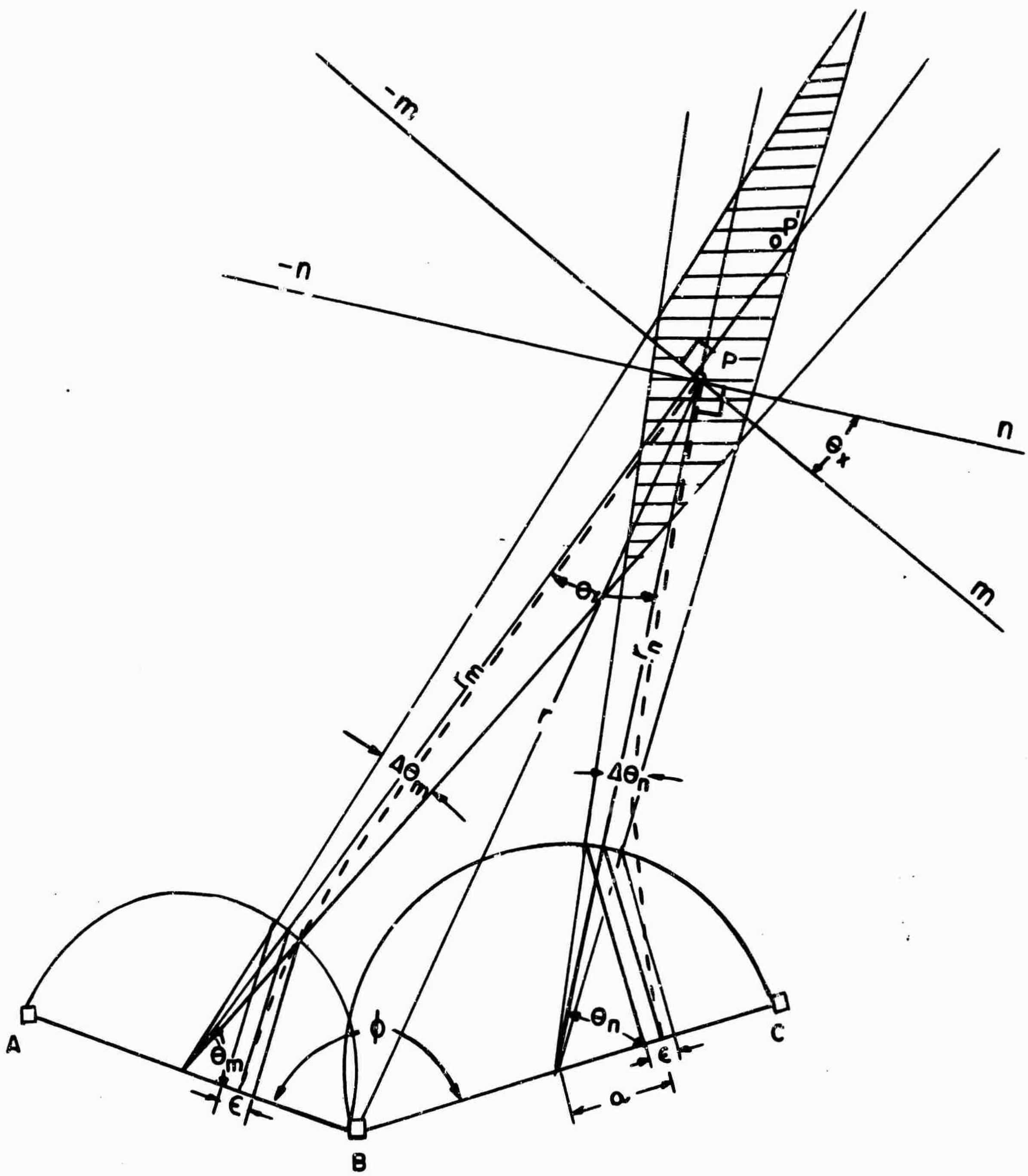

Figure 2. Basic Error Geometry (Three Observers)

or, perhaps more simply, starting with the original hyperbolic lines, the constants of difference would be obtained from

$$D_A - D_B = R(\theta_2 - \theta_1) = \text{constant} \tag{3}$$

It can be shown that these modified hyperbolic surfaces will intersect the earth in such a manner as to form closed curves which are ellipses, if four observers form two base lines which are conjugates of each other. The extension of such hyperbolas into ellipses on a spherical surface has been noted by Dr. E. A. Lewis of AFCRL.[1]

The various formulas which have been developed for the direct calculation of position from two or more pairs of time differences are quite complicated. None of these appears to offer the accuracy required for most applications. One of the most promising methods currently in use to obtain accuracy is: obtain an approximation of position using a relatively simple formula, calculate or look up the time differences which would occur if this were the true fix, and then compare these time differences with the data sets. The resulting so-called space/time error could be reduced to as small a value as desired by moving the position a small amount in an indicated direction and continuing iterations indefinitely. There is usually, however, a practical limit to the number of such iterations allowable.

Unfortunately, the error encountered in this calculation is not the only or worst error. It is the TOA measurement itself which is more fundamental.

What are the error distributions and how do they affect system engineering and total system accuracy? For a given timing error or timing error distribution, calculate geometrical areas which contain the measured point with a given probability.

Figures 2 and 3 relate the basic elements of the geometry involved. For a first order of accuracy and simplification of the model, it is assumed that we are dealing only with the plane surface representation of the total hyperbolic loci and that the propagations are straight lines in that plane.

The observers are A, B, C and A, B, C, D for the three- and four-observer configurations, respectively. Also the lines AB, BC and AB, CD, respectively, are the base lines of the three- and four-observer configurations.

The dashed curves represent the hyperbolas of a constant difference of time (or distance) as measured by these observers, and selected to include the point P. The

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[1]E. A. Lewis, Geometry and First-Order Error Statistics for Three- and Four-Station Hyperbolic Fixes on a Spherical Earth, AFCRL-64-461, June 1964.

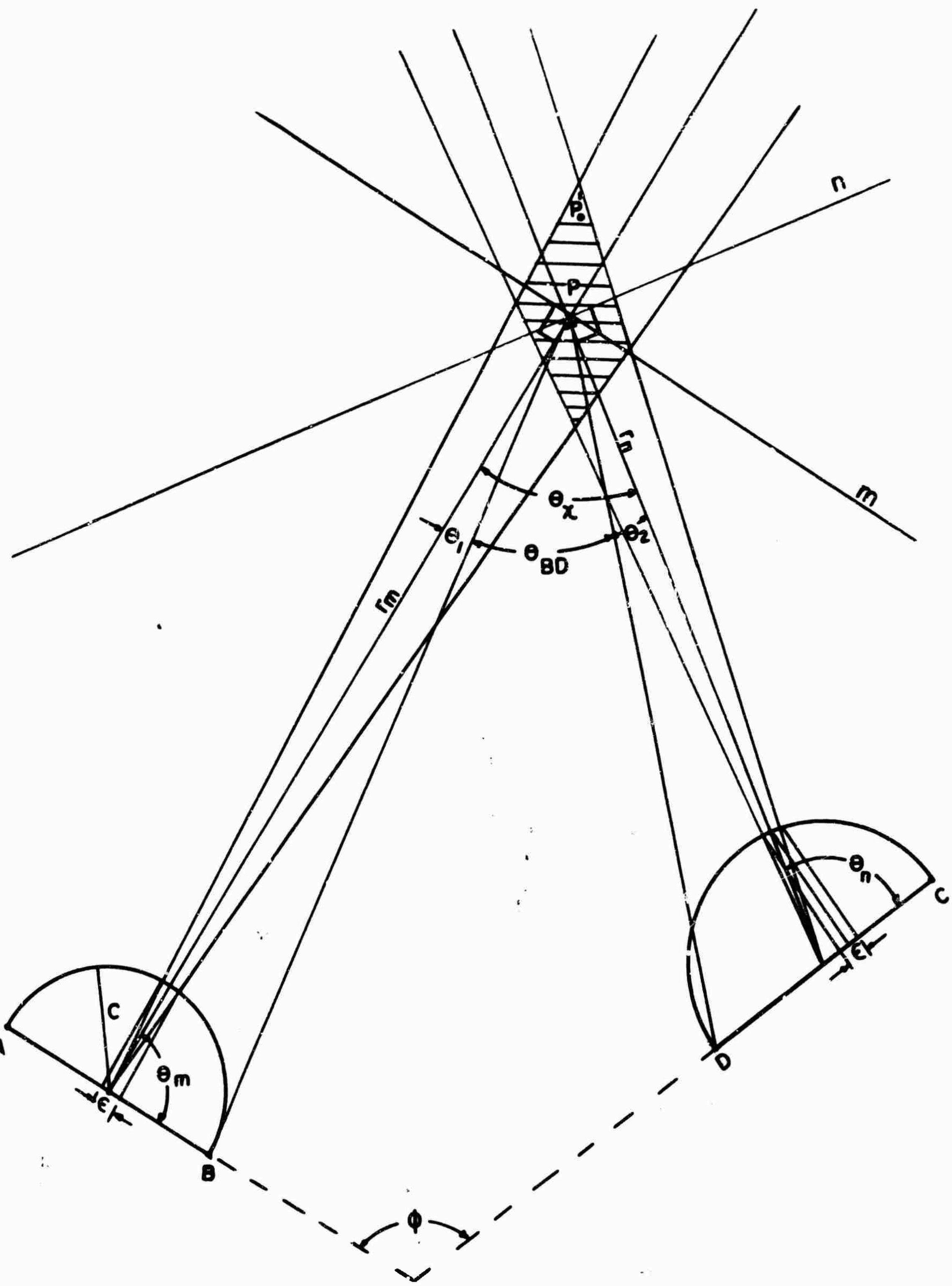


Figure 3. Basic Error Geometry (Four Observers)

asymptotes to these curves are represented as the straight lines from the center of each base line through the point P, thus introducing a small, negligible error in the angles, $\theta_{m,n}$. The exact construction of the asymptote is accomplished as follows:

a. The base intercept of the hyperbola is obtained from

$$2a = v\,(T_A - T_B) \tag{4}$$

b. A semicircle is drawn about each base line as a diameter, i.e., AB, BC.

c. A vertical line is erected from the base line at point a to the semicircle (Figure 2).

d. A line is drawn from the base line center to this vertical intercept of the semicircle, and then extended out as the asymptote.

The concept of the base line error, ϵ, is that the quantity $t_A - t_B$, will have errors $\pm\Delta t$ associated with it, also associated at the base line is a quantity, ϵ, given by

$$\epsilon = \frac{v\,\Delta t}{2}$$

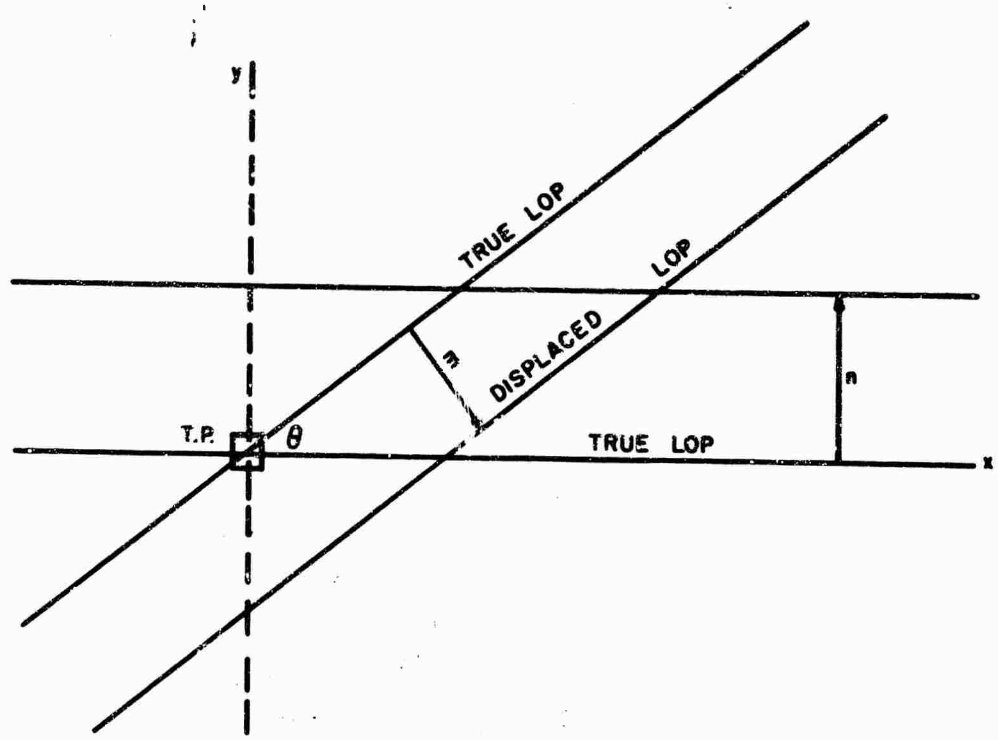
If, then, the asymptotes r_m, θ_m and r_n, θ_n have associated with them the errors $\Delta\theta_m$, $\Delta\theta_n$ which result from errors ϵ_m and ϵ_n, the crosshatched area would represent the area of uncertainty for given error limits. The equal division or statistical centering of $\Delta\theta$ about the true asymptotes is implied only for the sake of deriving a correction factor (see Appendix K). If $\Delta\theta_m$ and $\Delta\theta_n$ represented error limits, the crosshatched area would represent the area of uncertainty for those limits. In the sections to follow, except for one portion which actually deals with such a hard limited case, the size of θ, and hence $\Delta\theta$, is assumed to follow a normal probability distribution. Another assumption made is that, for purposes of engineering accuracy, the crosshatched area is a parallelogram bounded by four displaced hyperbolic LOP's (lines of position).

In order to measure or evaluate these LOP displacements, the oblique coordinate system m, n is established at right angles to r_m and r_n, respectively, since it is assumed the LOP's are parallel to the respective r_m or r_n.

Comparing the use of this oblique coordinate system to the usual orthogonal, rectangular coordinate system is believed to be unique in that it appears to be the first time that the complete analysis has been performed on the natural coordinates of the system. Secondly, it offers a certain advantage in eliminating the necessity for understanding, defining, and computing, variances, standard deviation, correlation coefficient, etc., required for the use of rectangular coordinates. To be more analytic, assume that we are dealing with the rectangular coordinate system x, y such that for a true fix point and given LOP displacements, for example m and n, we would have,

$$y = n$$

$$x = m \csc\theta + n \cot\theta$$



It then becomes necessary to establish a correlation coefficient between x and y. First establish the variance in x and variance in y. For the case of m and n, independent of each other, the situation is not too bad, with,

$$\sigma_y^2 = \sigma_m^2$$

$$\sigma_x^2 = \csc^2 \theta \, \sigma_m^2 + \cot^2 \theta \, \sigma_n^2$$

and correlation coefficient ρ_{xy} given by

$$\frac{COV(x, y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)\,E(y)}{\sigma_x \sigma_y}$$

which turns out to be

$$\rho_{xy} = \frac{COT\, \theta\, \sigma_n}{\sqrt{CSC^2\, \theta\, \sigma_m^2 + COT^2\, \theta\, \sigma_n^2}}$$

Having determined these fundamental rectangular parameters, the next step of comparison would be to integrate the probability density expression

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}\exp\left[-\frac{1}{2(1-\rho_{xy}^2)}\left(\frac{x^2}{\sigma_x^2}-\frac{2xy\,\rho_{xy}}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right)\right]$$

over the area of interest. Setting the exponent equal to a constant, such as C^2, results in the area being an ellipse with probability

$$\alpha = 1-e^{-C^2}$$

which agrees with the results obtained using the oblique system.

At this point perhaps the most significant difference is in determining the size of the ellipses for a given probability; the most significant advantage of the rectangular coordinates is that examination of the equation of the ellipse can provide the orientation and size of ellipse. By comparison, using oblique coordinates, a differential max/min determination must be made to obtain the true spatial description of the ellipses.

Referring to the rectangular parameters, when the quantities m and n are not independent (three observers), the determination of variance x and the resulting ρ_{xy}, are considerably more complicated, resulting in much more complex expressions for the ellipses.

By comparison, measurement and analysis in the oblique system virtually eliminate the requirement for establishing these statistical parameters in the space domain. As shown in this report, once the correlation factor of errors in the paired time differences (Δt_n and Δt_m) is determined, and the density function, $f(\Delta t_n, \Delta t_m)$, is known the conversion to the spatial representation follows without further statistical analysis.

Before comparing the difficulties of determining a circular area which contains the point with a probability α, a Quasi-Circular Error Probability (QCEP) is defined. A circular area of radius, R, regardless of the coordinate system, oblique or otherwise, which contains the point in question with a probability α is determined. This differs from the classical definition of CEP[3] in two respects. First, values of R for all values of α from 0 to 1 are obtained, rather than the special case of $\alpha = 0.5$ only. Second, the variates, whether x, y coordinates or m, n coordinates, are not equal nor does $R=\sqrt{m^2+n^2}$; hence, there may or may not be a Rayleigh function involved, so it is 50% probable that R exceeds 1.177 of some equivalent σ. Nonetheless, a joint probability function throughout a circle is integrated and the results are abbreviated QCEP.

[3] Reference Data for Radio Engineers, International Telephone and Telegraph Corporation, 4th Edition.

Contrary to calculations of the QCEP in the oblique system (Appendices C and D) complication rather than simplification results from utilizing a rectangular coordinate system.

In the case of four observers, the rectangular coordinate system requires a solution of

$$\alpha = f_1(\sigma_x, \sigma_y, \theta_x) \int_{-R}^{R} \left\{ \Phi[f_2(n, \sigma_x, \sigma_y, \theta_x)] - \Phi[f_3(n, \sigma_x, \sigma_y, \theta_x)] \right\} \exp\left[-f_4(n^2, \sigma_x, \sigma_y, \theta_x)\right] dn$$

where

$$f_1 = \frac{1}{4\sigma_x \sigma_y \sqrt{1-\rho_{xy}^2}}$$

$$f_2 = \frac{Dn}{2A} + AR$$

$$f_3 = \frac{Dn}{2A} - AR$$

$$f_4 = (A - \frac{D}{4A^2})n^2$$

and

$$A = \frac{1}{2\sigma_x \sigma_y (1-\rho_{xy}^2)}$$

$$D = \frac{\rho_{xy}}{\sigma_x \sigma_y (1-\rho_{xy}^2)}$$

$$\sigma_y = \sigma_n$$

$$\sigma_x = \sqrt{\mathrm{CSC}^2\, \theta_x\, \sigma_m^2 + \mathrm{COT}^2\, \theta_x\, \sigma_n^2}$$

$$\rho_{xy} = \frac{\mathrm{COT}\ \theta_x\, \sigma_n}{\sqrt{\mathrm{CSC}^2\, \theta_x\, \sigma_m^2 + \mathrm{COT}^2\, \theta_x\, \sigma_n^2}}$$

whereas, in the oblique system, the calculation

$$\sigma = \frac{1}{2\sqrt{2\pi}\,\sigma_n} \int_{-R}^{R} (\Phi[v_1] - \Phi[v_2]) \exp\left[-\frac{n^2}{2\sigma_n^2}\right] dn$$

where

$$v_1 = \frac{\sin\theta_x}{\sqrt{2}\,\sigma_m} (n \cot\theta_x + \sqrt{R^2 - n^2})$$

$$v_2 = \frac{\sin\theta_x}{\sqrt{2}\,\sigma_m} (n \cot\theta_x - \sqrt{R^2 - n^2}) .$$

For the case of three observers, the ratio of complexity of rectangular over oblique systems is considerably greater. Both systems will add a term to the Φ functions, but for the rectangular coordinates there is, in effect, a correlation factor within a correlation factor, which modifies the parameters as follows

$$\sigma_y = \sigma_n$$

$$\sigma_x^2 = \csc^2\theta_x\,\sigma_m^2 + \cot^2\theta\,\sigma_n^2 + \csc\theta_x \cot\theta_x\,\sigma_n\sigma_m$$

$$\rho_{xy} = \frac{\frac{1}{2}\csc\theta_x\,\sigma_m\sigma_n + \cot^2\theta_x\,\sigma_n^2}{\sigma_n\sqrt{\csc^2\theta_x\,\sigma_m^2 + \cot^2\theta_x\,\sigma_n^2 + \csc\theta_x\cot\theta_x\,\sigma_m\sigma_n}}$$

whereas, for the oblique system, the functions are changed only by the addition of a minus $n/\sqrt{3/2}\,\sigma_n$ term.

As shown in Appendix A, the angles which form the asymptotes to the hyperbolas are given by

$$\theta_n = \cos^{-1}\left|\frac{v(t_B - t_C)}{2C}\right| \tag{5}$$

$$\theta_m = \cos^{-1}\left|\frac{v(t_A - t_B)}{2C}\right| \tag{6}$$

For all points beyond a distance of one base line, these asymptotes are a very good approximation to the hyperbolas themselves, e.g.,

$$r_n \text{ and } r_m > 2C \tag{7}$$

If greater accuracy is required for fixes which are closer, either more precise formulas must be used or solution by construction can be performed. The latter may be performed by referring to a preconstructed system of hyperbolas, provided the incremental steps in time differences per hyperbola are fine enough. Paren - thetically, and as a philosophic rationalization, it is stated that the utility of this model is proportional in some manner to the range. That is, it is difficult to visualize applications where r_n or r_m are less than 2C. One probably encounters such things as: "I can see the guy with my naked eye, "or, " If the person or the event is that close to home, who needs a system to measure its location?"

When θ_n and θ_m have been determined, these and the crossing angle θ_x are related by

$$\theta_x = 180^\circ + \theta_n - (\phi + \theta_m) \tag{8}$$

where ϕ is the angle between the base lines. Perhaps the simplest way to determine the distances r_n and r_m is by construction on the plane representation of the base line system.

If direct calculation is preferred, it may be obtained as follows:

In Figure 4, since the line d joining the centers of the fixed base lines is fixed, the angles e and f are also fixed. Then using the law of sines

$$\frac{d}{\sin \theta_x} = \frac{r_m}{\sin (180-f-\theta_n)} = \frac{r_n}{\sin (\theta_m - e)} \tag{9}$$

$$r_m = d \frac{\sin (180-f-\theta_n)}{\sin \theta_x} \tag{10}$$

$$r_n = d \frac{\sin (\theta_m - e)}{\sin \theta_x} \tag{11}$$

These relations are equally applicable to either the case of three observers or four observers. Thus we have performed either a calculation or graphical solution, or combination, for all the geometric variables associated with a given set of fix data

Regarding accuracy in the use of plane geometry, using the spherical surface of the earth, either a scaled sphere or globe or a plane map projection of the earth may be used for the construction and measurement of the geometric factors. The former is apt to be quite inconvenient. Concerning the latter, if the map were distortionless, and straight lines drawn on the map were the true representation of great circle segments of the globe, then the base lines and asymptotes would be true, and the plane model would be without error.

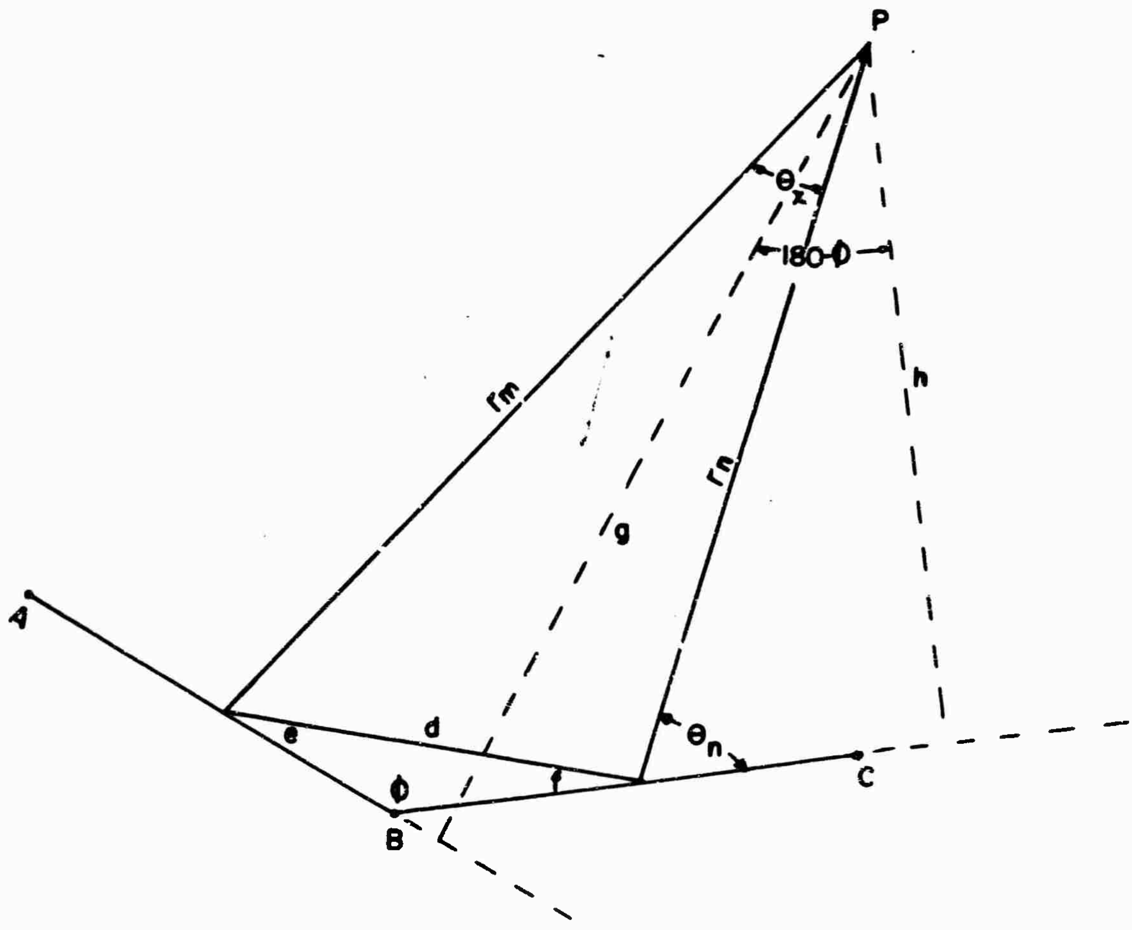


Figure 4. Geometry of Angles Involved

Since a Lambert conformal conic projection contains a distance measurement maximum error of about 1% to 2% for the ranges of concern herein, and since we are for most cases utilizing the asymptotes as LOP's (lines of position), the errors should be negligible. This is particularly true if the crossing angle, θ_x, is calculated directly in terms of Lat.-Long. and spherical angles. An enabling factor is the axiom that the tangent to the hyperbola at the point P bisects the angle subtended by the base line, such as APB. Thus, if we are seeking a statistical analysis for some assumed point, P, it is relatively simple to establish the angle, θ_1 and θ_2 (see Figure 3) of the true hyperbolas and then obtain

$$\theta_x = \theta_1 + \theta_2 + \theta_{BD}$$

where, for three observers, $\theta_{BD} = 0$

The details and derivation of such quantities as n, m, θ_n, θ_m, R and Γ are given in Appendix A. As seen in Figure A-1, determination of these quantities is independent of the number of observers involved since we are dealing with only one given pair of readings at a time. In resumé, the formulas for these quantities follow:

$$n = \Gamma_n \Delta t_n \,;\, m = \Gamma_m \Delta t_m \qquad (12)$$

$$\theta_n = \cos^{-1} \frac{v(T_B - T_C)}{2C} \qquad (13)$$

$$R^2 = \frac{1}{\sin^2 \theta_x} (m^2 + n^2 - 2mn \cos \theta_x) \qquad (14)$$

where, in addition to the quantities previously defined, Δt_n is the net timing error made by A and B after subtraction, and Δt_m is the error of B and C, or C and D.

Γ_n is a time to space gradient whose value is given by

$$\Gamma_n = \frac{r_n v}{2C \sin \theta_n} \qquad (15)$$

θ_x is the crossing angle formed by the intersecting hyperbolas or their asymptotes.

R is the distance in the oblique axis system m, n to any point P' from the origin P.

C is the base line radius or one-half of the base line.

This concludes the essential geometry of the fix system.

III. DISCUSSION OF UNIFORM PROBABILITY OF ERROR GEOMETRY

The probability of committing an error of any magnitude has not yet been mentioned. Of the many possible distributions of errors, this discussion shall be limited to two:

1. Equal or uniform probability,
2. Normal distribution of error.

The first case will be dealt with rather briefly. While such a distribution may never occur, its analysis has academic value if for no other reason than to provide a comparison or check against results of the second case or, for that matter, any other distribution. The details of this analysis are given in Appendix B. The important result to be noted is that for given limits of error of ϵ the radius of a circle which would contain one-half of all the measured fixes, i.e., the QCEP (quasi-circular error probability) is closely approximated by the following expressions:

CASE	RADIUS R	
$2R < \Delta S_n < \Delta S_m$	$R = \epsilon/C\sqrt{\dfrac{r_n r_m}{2\pi \sin\theta_x \sin\theta_n \sin\theta_m}}$	(16)
$\Delta S_n < 2R < \Delta S_m$	$R = \epsilon/C\,\dfrac{r_n}{4C \sin\theta_x \sin\theta_n}$	(17)
$\Delta S_n < \Delta S_m < 2R$	$R = \dfrac{\epsilon}{2C}\sqrt{\dfrac{r_m^2}{4\sin^2\theta_x \sin^2\theta_m} + \dfrac{r_n^2}{\sin^2\theta_n}}$	(18)

The QCEP, for the cos exp α = 50%, means that there is a 50% probability that any one particular calculation using data which contains errors will produce a point P which lies within this same circle. It can be seen from Figures 2 and 3, and from the previous discussion, that there is nothing inherently natural about a circular area of fix points. In fact, if a continuous smooth curve of equal error probability were required to approximate this area, it would be inclined to be an ellipse. The inherent benefit of elliptical contours will be seen in the second case of normal distribution of error. Meanwhile, we are faced with a necessity of at least being able to compute QCEP's in order to be compatible with other systems which cause the event P to occur, where the probability of cause is inherently circular.

Since the second case of a normally distributed probability of committing an error ϵ is believed to be more representative of this system, it will be treated in considerably more detail. One example of a component of timing which may depart from the normal distribution of variance is the velocity of propagation of the event. Actually, throughout this study, the velocity is assumed to be constant, i.e., given no distribution whatsoever. However, if the magnitude of variance, with different paths, is significant, it is believed that its distribution is more apt to be Rayleigh than normal.

IV. DETERMINATION OF STANDARD DEVIATIONS OF LINES OF POSITION DISPLACEMENTS

This section discusses the determination of σ_n and σ_m, the standard deviations of the statistical displacements of P' as measured along the axes, n, m.

The first step is to establish the probability density distribution of the timing errors for a pair of observer measurements. This is done in Appendix E.

The significant features are:

If σ = standard deviation of timing errors encountered at each observer station

T_A = exact true TOA at station A

T_B = exact true TOA at station B

The probability density function of timing errors at A and B are

$$f(t_A) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t_A - T_A)^2}{2\sigma^2}\right] \tag{19}$$

$$f(t_B) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t_B - T_B)^2}{2\sigma^2}\right] \tag{20}$$

and the density function for the pair is

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{(M-x)^2}{2\sigma_x^2}\right] \tag{21}$$

where

$$x = t_A - t_B \text{ or } t_B - t_C$$

$$M = T_A - T_B \text{ or } T_B - T_C$$

$$\sigma_x = \sqrt{2}\,\sigma$$

This also defines the important quantity σ_x.

For convenience and reduction of terms, the quantity M-X and the quantities Δt_n and Δt_m as used in Appendix A and elsewhere are the same. Thus, to clarify and standardize nomenclature, the density functions for each pair are written

$$f(t_A - t_B) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{\Delta t_m^2}{2\sigma_x^2}\right] \tag{22}$$

or

$$\left.\begin{array}{l} f(t_B - t_C) \\ f(t_D - t_C) \end{array}\right\} = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[- \frac{\Delta t_n^{\;2}}{2\sigma_x^{\;2}} \right] \tag{23}$$

Referring again to Appendix A, and recalling that

$$\Gamma_n = \frac{v\, r_n}{2C \sin \theta_n} \tag{24}$$

$$\Gamma_m = \frac{v\, r_m}{2C \sin \theta_m} \tag{25}$$

therefore

$$n = \Gamma_n \, \Delta t_n \tag{26}$$

$$m = \Gamma_m \, \Delta t_m \tag{27}$$

It is again emphasized that this engineering simplification degrades rapidly in accuracy as the distances r_n and r_m become less than 2C and/or as the angles θ_n and θ_m approach 0^o or 180^o.

Consider the special value

$$\Delta t_n = \sigma_x \tag{28}$$

giving the special value

$$n_o = \Gamma_n \, \sigma_x \tag{29}$$

Using standard notation for the normal density function for variations in the value of n gives

$$f(n) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left[- \frac{n^2}{2\sigma_n^{\;2}} \right] \tag{30}$$

Substitution gives

$$f(n_o) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left[- \frac{\Gamma_n^{\;2} \sigma_x^{\;2}}{2\sigma_n^{\;2}} \right] \tag{31}$$

If n_0 is interpreted as that value of n which makes $f(n) = 0.242/\sigma_n$, i.e., the one sigma value, then

$$\Gamma_n^2 \sigma_x^2 = \sigma_n^2 \tag{32}$$

and we have the important relationships

$$\sigma_n = \Gamma_n \sigma_x \tag{33}$$

$$\sigma_m = \Gamma_m \sigma_x \tag{34}$$

A formula which gives a closer approximation for values of r_n or $r_m < 2C$ is obtained from Dr. E.A. Lewis[10] and B.W. Sitterly *

$$\sigma_m = \frac{\tau_{AB} \times v}{2 \sin \frac{B_A - B_B}{2}} = \sigma_x \frac{v}{2 \sin \frac{B_A - B_B}{2}} \tag{33a}$$

where

"τ_{AB} is the standard deviation of the errors in time-difference measurement made by stations A and B," and is therefore equivalent to σ_x . B_A and B_B are the respective bearings of the lines AP and BP measured with respect to the true north vector at P. Thus $B_A - B_B$ is the base line angle APB subtended at P.

In fact, if we consider the earth spherical and can measure B_A and B_B more readily and/or accurately than r and θ , this computation of σ_m and σ_n where

$$\sigma_n = \frac{\sigma_x v}{2 \sin \frac{B_C - B_B}{2}} \tag{34a}$$

may be preferred to the gradient $\Gamma_{n/m}$ for

$$\sigma_{n/m} = \sigma_x \Gamma_{n/m}$$

used in this study.

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*Loran, MIT Radiation Laboratory Series, Vol 4, Mc Graw-Hill, New York, 1947.

A more general proof of the validity of this concept follows. Using the probability theorem that if

$$v = f(u)$$

$$p(v) = p(u)\left|\frac{du}{dv}\right|$$

we have

$$n = f(\Delta t_n)$$

$$n = \Gamma_n \Delta t_n \qquad (35)$$

or

$$\Delta t_n = \phi(n) = \frac{n}{\Gamma_n} \qquad (36)$$

hence

$$\Delta t_n = p(n)\left|\frac{dn}{dt}\right| \qquad (37)$$

but

$$\frac{dn}{d\Delta t_n} = \Gamma_n$$

so

$$p(\Delta t_n) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left[-\frac{n^2}{2\sigma_n^2}\right] \Gamma_n$$

$$p(\Delta t_n) = \frac{1}{(\sigma_n/\Gamma_n)\sqrt{2\pi}} \exp\left[-\frac{\Gamma_n^2 \Delta t_n^2}{2\sigma_n^2}\right] \qquad (38)$$

$$p(\Delta t_n) = \frac{1}{(\sigma_n/\Gamma_n)\sqrt{2\pi}} \exp\left[-\frac{\Delta t_n^2}{2\sigma_n^2/\Gamma_n^2}\right]$$

and, if

$$\sigma_x = \frac{\sigma_n}{\Gamma_n} \qquad (39)$$

this becomes the identity

$$p(\Delta t_n) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{\Delta t_n^{\,2}}{2\sigma_x^{\,2}}\right] \tag{40}$$

By symmetry

$$\sigma_x = \frac{\sigma_m}{\Gamma_m} \tag{41}$$

Another relationship which introduces the important parameter, β, is

$$\frac{\sigma_n}{\sigma_m} = \frac{\Gamma_n \sigma_x}{\Gamma_m \sigma_x} = \frac{\Gamma_n}{\Gamma_m} = \beta \tag{42}$$

Further

$$\frac{\Gamma_n}{\Gamma_m} = \frac{\dfrac{v\, r_n}{2c \sin \theta_n}}{\dfrac{v\, r_m}{2c \sin \theta_m}} = \frac{r_n \sin \theta_m}{r_m \sin \theta_n} \tag{43}$$

If g and h are the vertical components of r_n and r_m onto the base lines and if $r_n \simeq r_m$ then for a quick approximation

$$\frac{\sigma_n}{\sigma_m} = \frac{\Gamma_n}{\Gamma_m} = \frac{g}{h} = \beta \tag{44}$$

This approximation is valid for the geometry of both the three and four observers, and should complete the basic model of the system probability/geometry.

V. DETERMINATION OF QUASI-CIRCULAR ERROR PROBABILITY FOR THE CASE OF FOUR OBSERVERS

From Figures 2 and 3, calculate the probability that a given set of time measurements will result in a fix which is displaced from its true position P to P'. P' is measured in the oblique coordinate system m, n as the distance to the vertical projections of P' on n and m. This is also representative of the displacements of the hyperbolic LOP's from the true LOP's.

At this point the QCEP analysis must separate into two branches; as will be shown, there is significant difference between the probabilities, and hence radii, for the case of four versus three observers.

The case of four observers is treated first because of the simplification that the paired differences of time readings are independent of each other, whereas such is not the case for three observations.

If p(n) and p(m) represent the respective probabilities of displacements of magnitude n and m, and if the environment and readings of A and B are entirely independent of C and D, then

$$f(n, m) = f(n) \times f(m) \tag{45}$$

Also, for normal probability density distribution, we may utilize from standard probability notation, the following:

$$f(n) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left[-\frac{n^2}{2\sigma_n^2}\right] \tag{46}$$

$$f(m) = \frac{1}{\sigma_m \sqrt{2\pi}} \exp\left[-\frac{m^2}{2\sigma_m^2}\right] \tag{47}$$

or,

$$f(n, m) = \frac{1}{\sigma_n \sigma_m 2\pi} \exp\left[-\left(\frac{n^2}{2\sigma_n^2} + \frac{m^2}{2\sigma_m^2}\right)\right] \tag{48}$$

The associated probabilities for incremental regions Δn and Δm then become

$$p(n) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left[-\frac{n^2}{2\sigma_n^2}\right]\Delta n \tag{49}$$

$$p(m) = \frac{1}{\sigma_m \sqrt{2\pi}} \exp\left[-\frac{m^2}{2\sigma_m^2}\right]\Delta m \tag{50}$$

Consider now the region A (which shall become a circle) and all the various combinations of unique points n, m and their probabilities of occurrence. Since these are mutually exclusive events, the total (collectively exhaustive) probability for the region A is given by

$$\alpha = \sum_m \sum_n f(n) \times f(m)\ \Delta m\ \Delta n \tag{51}$$

or, by well-known use of the integral calculus for the continuous case,

$$\alpha = \frac{1}{2\pi\ \sigma_n\ \sigma_m} \int\int_A \exp\left[-\frac{n^2}{2\sigma_n^2}\right] \exp\left[-\frac{m^2}{2\sigma_m^2}\right] dmdn \tag{52}$$

For the special (QCEP) case of the region A bounded by a circle of radius R

$$\alpha = \frac{1}{2\pi\ \sigma_m\ \sigma_n} \int_{-R}^{R} \int_{f_1(n)}^{f_2(n)} \exp\left[-\frac{n^2}{2\sigma_n^2}\right] \exp\left[-\frac{m^2}{2\sigma_m^2}\right] dmdn \tag{53}$$

where

$$f_1(n) = n \cos \theta_x - \sqrt{R^2 - n^2} \sin \theta_x \tag{54}$$

$$f_2(n) = n \cos \theta_x + \sqrt{R^2 - n^2} \sin \theta_x \tag{55}$$

A close approximation for small values of $R \sin \theta_x$ was found to be

$$\alpha = \frac{1}{4\pi\ \sigma_n\ \sigma_m} \int_{-R}^{R} [f_2(n) - f_1(n)] \left\{ \exp\left[-\frac{f_1^2(n)}{2\sigma_m^2}\right] + \exp\left[-\frac{f_2^2(n)}{2\sigma_m^2}\right] \right\} \exp\left[-\frac{n^2}{2\sigma_n^2}\right] dn \tag{56}$$

A more exact expression developed in Appendix C, which is good for all θ_x up to the practical limit of 90°, is given by

$$\alpha = \frac{1}{2\beta\sqrt{2\pi K}} \int_{-1}^{+1} [\Phi_1(x) - \Phi_2(x)] \exp\left[-\frac{x^2}{2K\beta^2}\right] dx \tag{57}$$

where

$$\Phi_1(x) = \Phi\left\{\frac{\sin\theta}{\sqrt{2K}}\left(x\cot\theta + \sqrt{1-x^2}\right)\right\} \tag{58}$$

$$\Phi_2(x) = \Phi\left\{\frac{\sin\theta}{\sqrt{2K}}\left(x\cot\theta - \sqrt{1-x^2}\right)\right\} \tag{59}$$

and the symmetry of results is such that solutions can be made in terms of that variable whose standard deviation is the greater. The tabulated results follow.

TABLE I

NOTATION AND VALUES OF QCEP VARIABLES

	If	
	$\sigma_n > \sigma_m$	$\sigma_m > \sigma_n$
x =	n/R	m/R
β =	σ_n/σ_m	σ_m/σ_n
K =	$\sigma_m{}^2/R$	$\sigma_n{}^2/R$

Figure 5 represents the results of the use of these equations to determine the circle R versus the required probability, α . The line drawn through the curves at α = 50% gives the required values of R, in terms of σ_n or σ_m, for the special QCEP equivalency case.

Note that β, θ_n, and θ_x appear related in some complex manner. If this is true, one of the parameters β or θ_x could possibly be eliminated thereby reducing the number of curves necessary to describe the system under these conditions. The fact, however, that β and θ_x are not entirely independent of each other does not invalidate their use as parameters. Values for these curves were computed on the CDC 1604B computer. Both formulas were used for small values of R sin θ with very good result comparison. To obtain values of the error functions for Φ_1 and Φ_2, a program was successfully written based on the infinite series given in Jahnke and Emde which resulted in good or better accuracy than the values in their table.

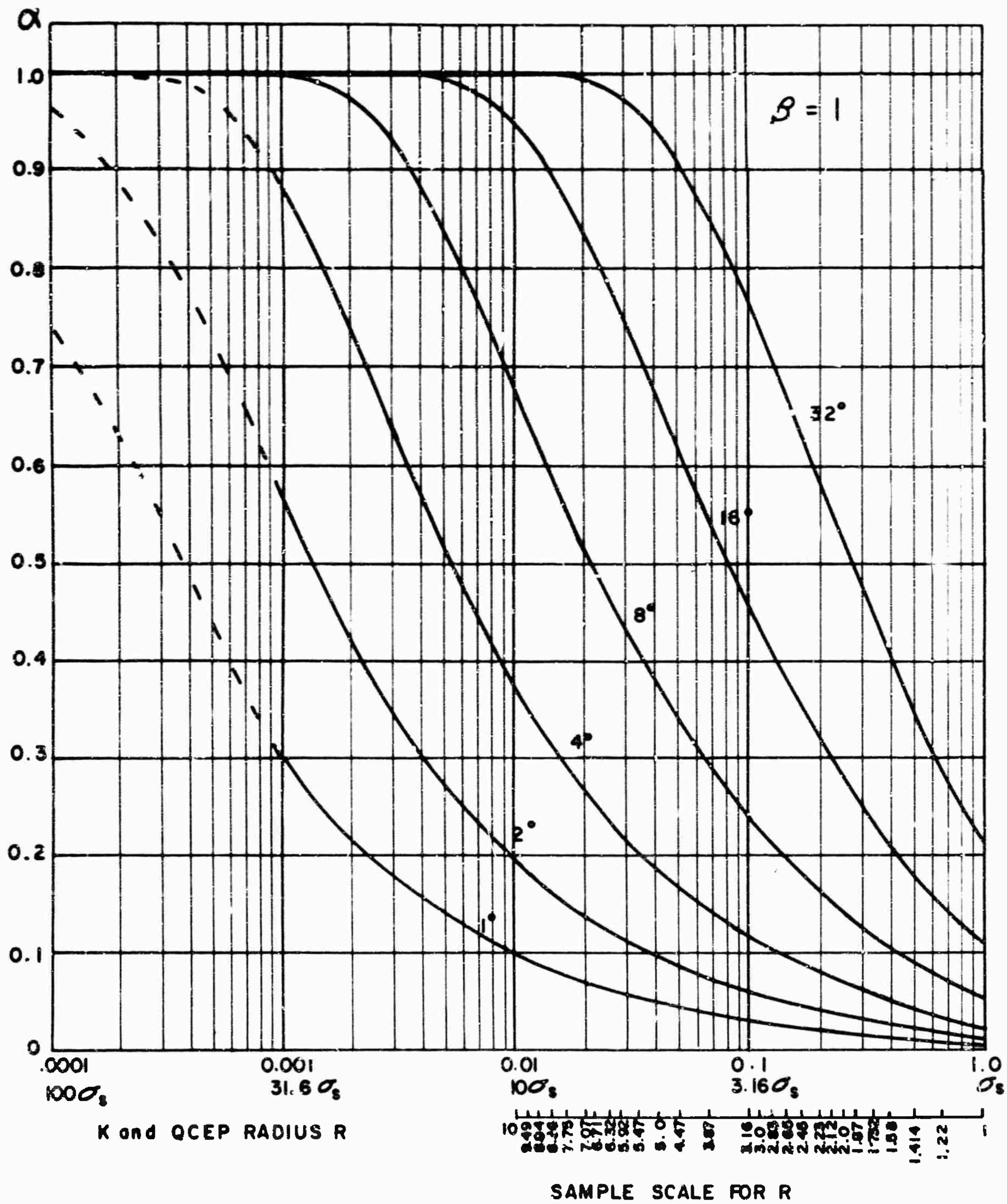


Figure 5(a). QCEP Radii Versus Probability α for the Case of Four Observers

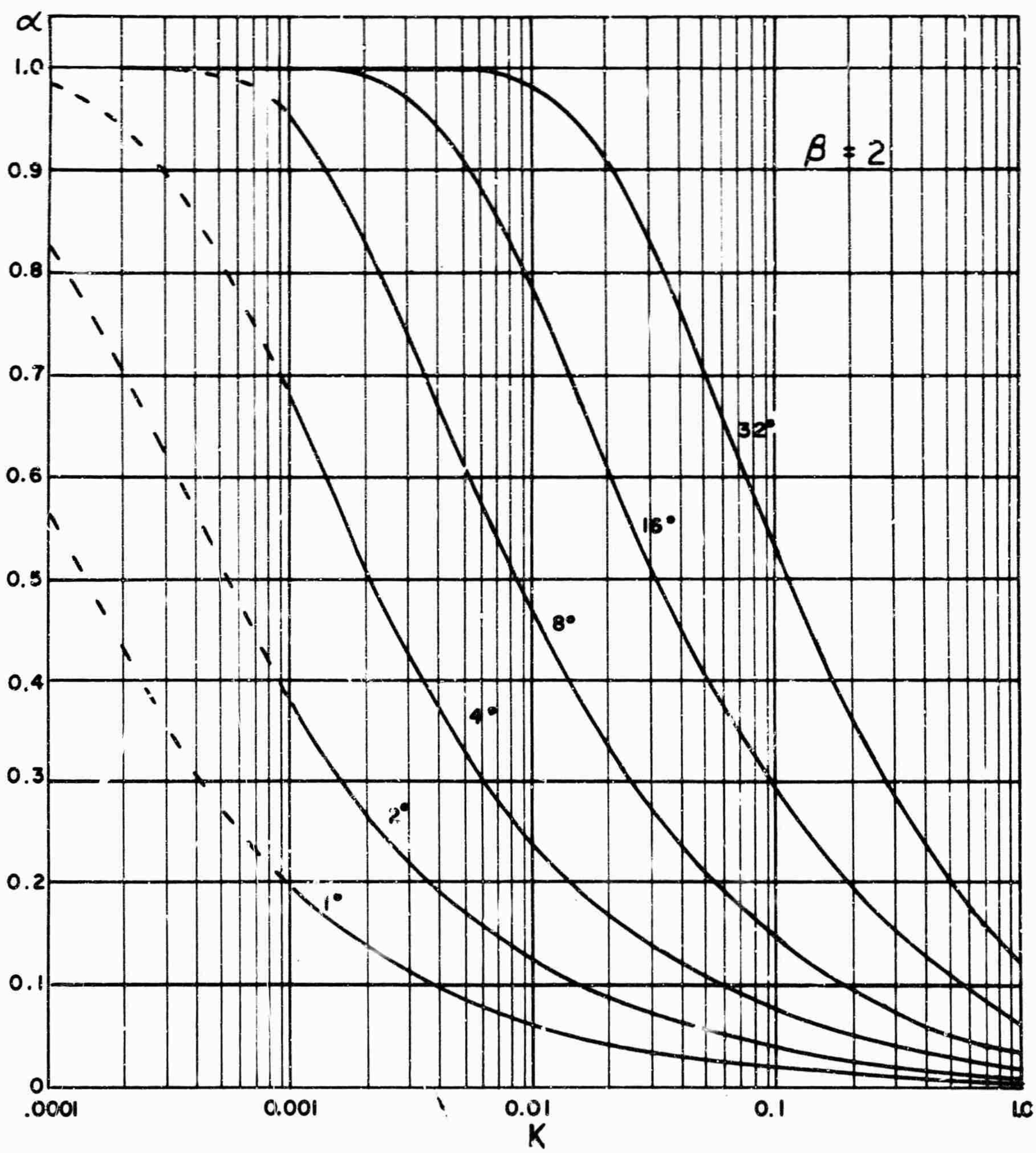


Figure 5(b). QCEP Radii Versus Probability α for the Case of Four Observers

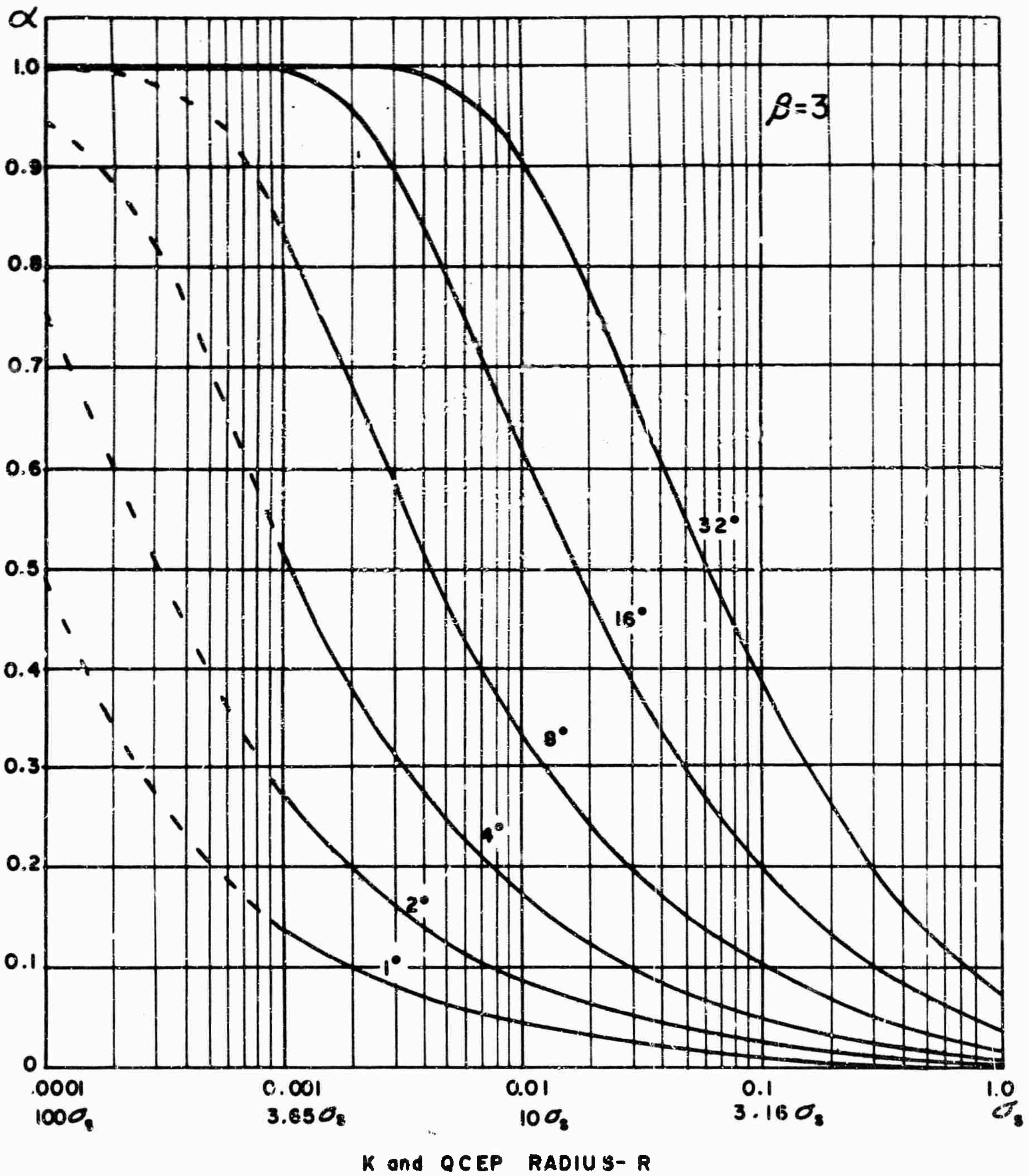


Figure 5(c). QCEP Radii Versus Probability α for the Case of Four Observers

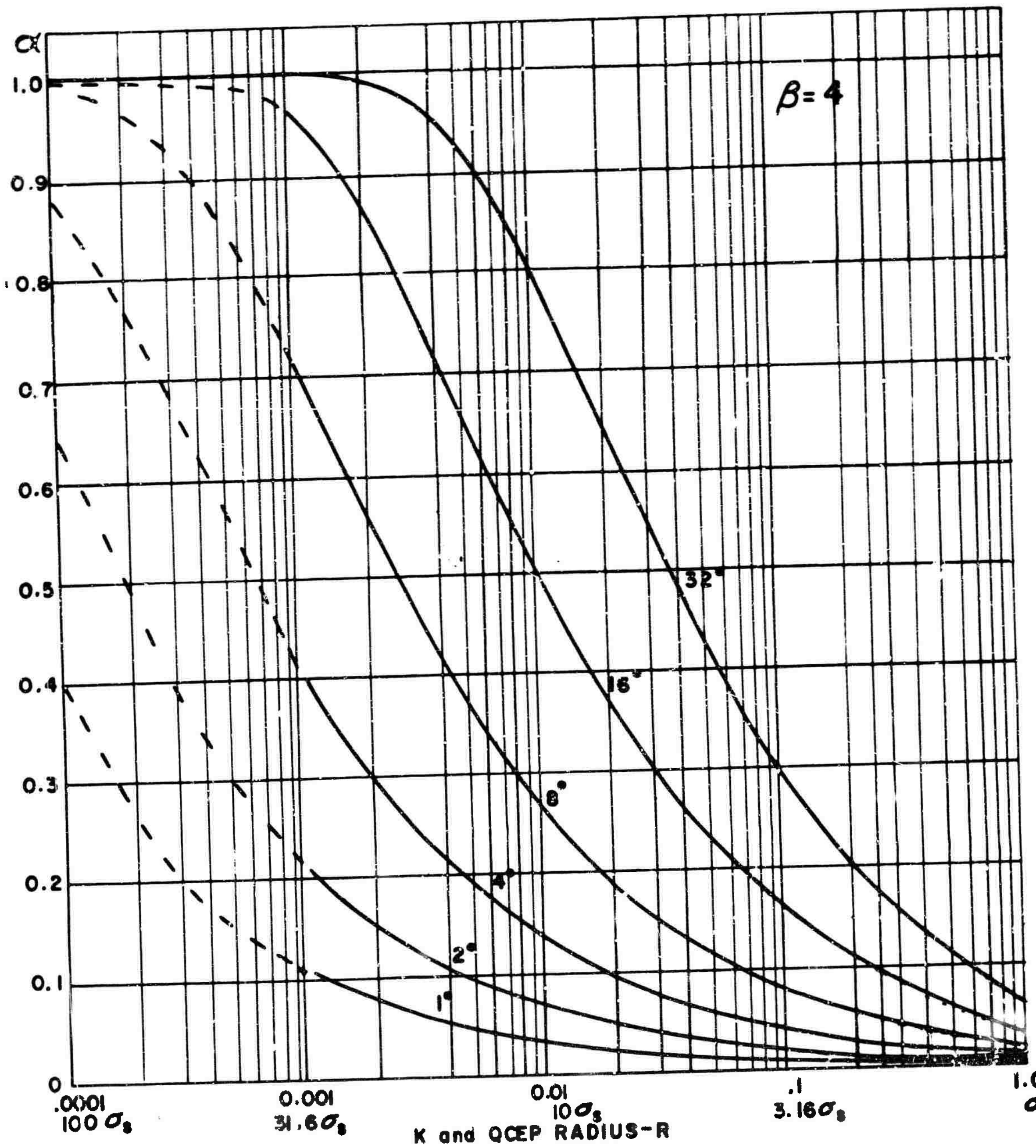


Figure 5(d). QCEP Radii Versus Probability α for the Case of Four Observers

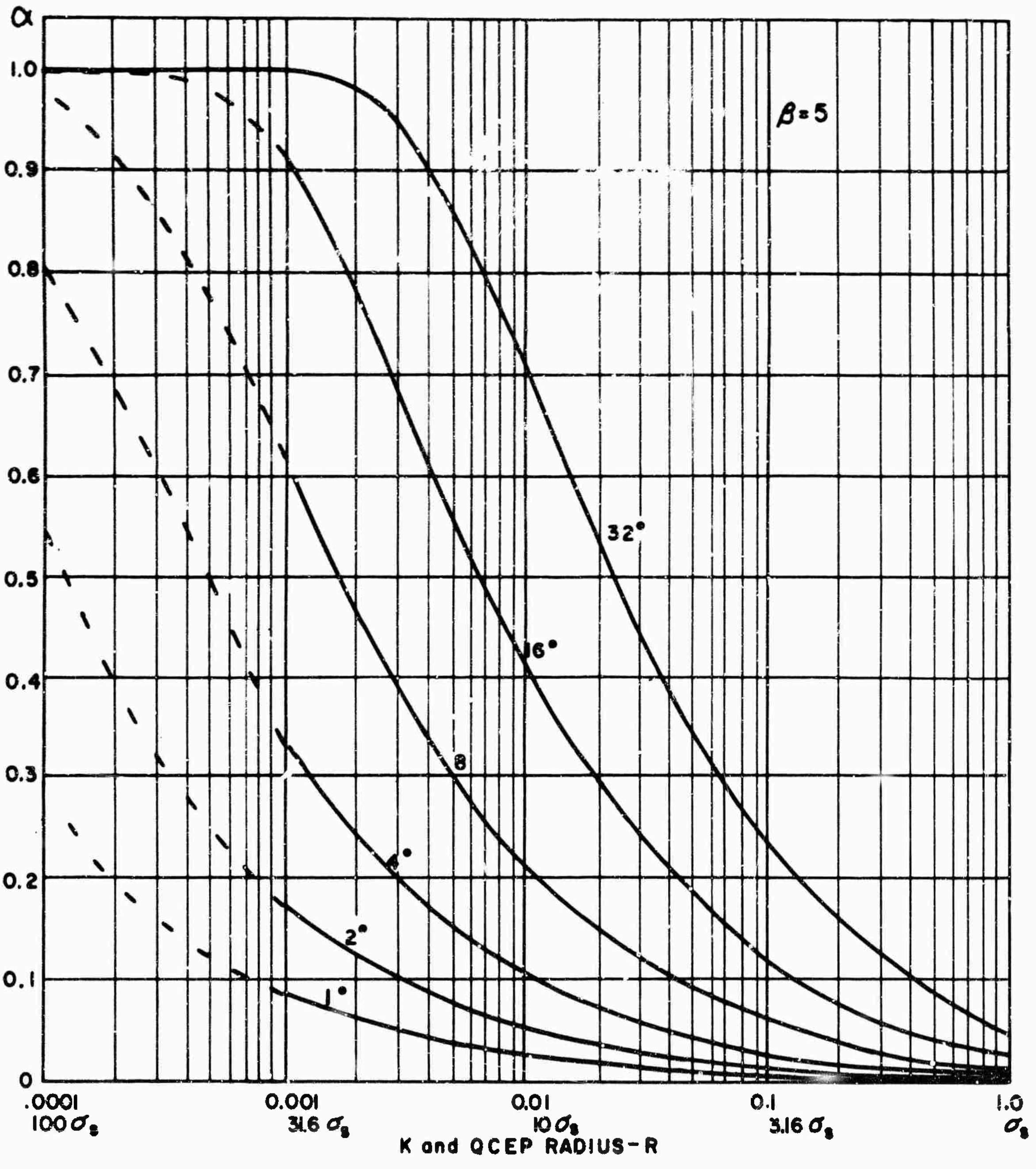


Figure 5(e). QCEP Radii Versus Probability α for the Case of Four Observers

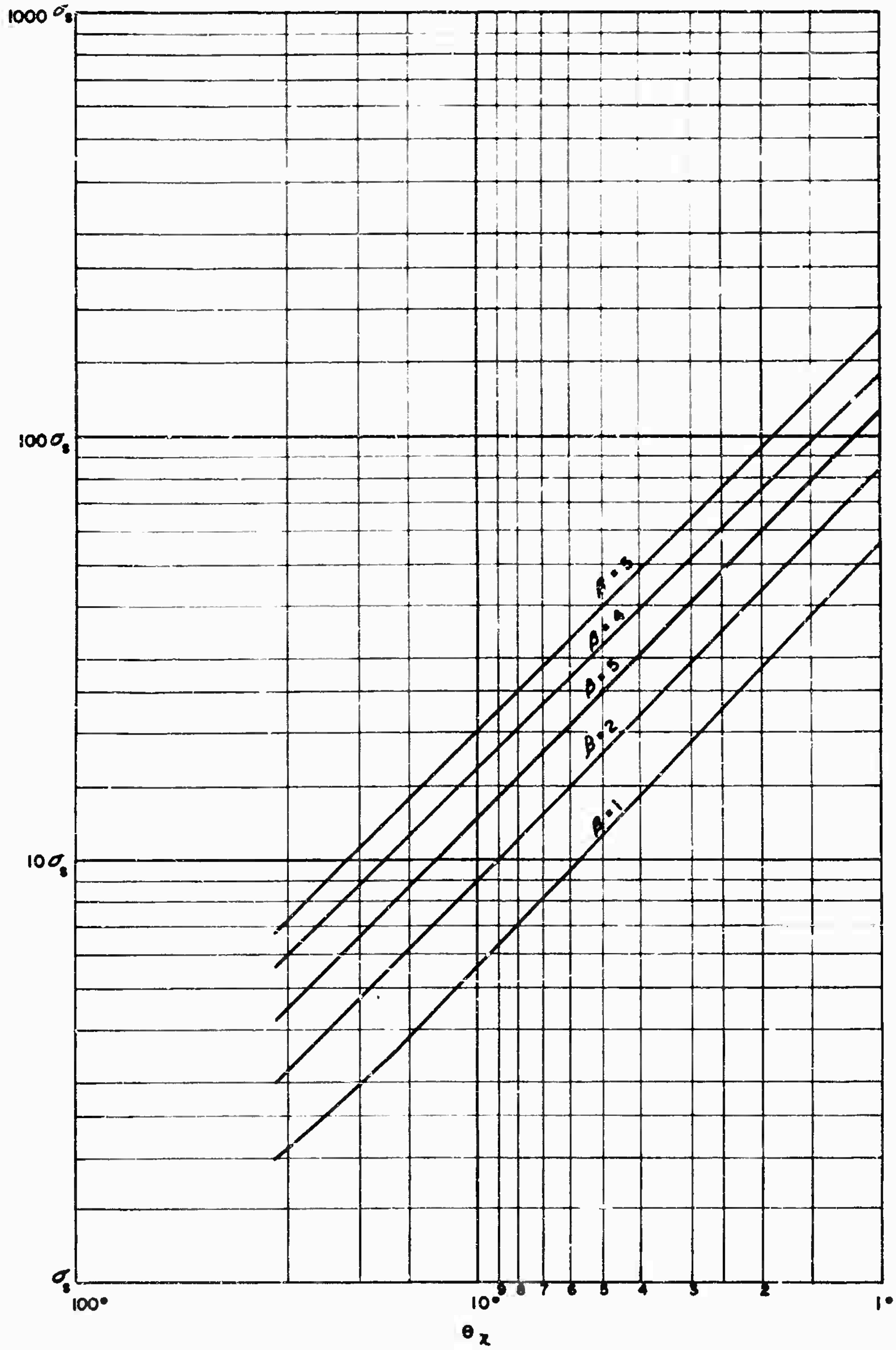


Figure 5(f). QCEP Radii Versus Probability α for the Case of Four Observers

VI. DETERMINATION OF THE QCEP FOR THE CASE OF THREE OBSERVERS

Compare the case of four observers with that of the three observers in Figure 2 where, in effect, observers B and D are merged into one. The situation with respect to the independence of pairs of time differences is now changed; they are no longer independent. In terms of a coefficient of correlation, or correlation factor, consider two theoretical extremes to help visualize the situation. First consider B to be errorless, with all of the Gaussian distributed errors committed at A and C. This would again represent an independent situation or one of zero correlation factor. Next consider A and C to be errorless, and assume all of the errors to be committed by B. This would represent a one-to-one dependency with a correlation factor of one. The fact that the three observers create errors with equal distribution makes the actual correlation coefficient of 0.5 seem intuitively feasible. The proof of $\rho = 0.5$ is given in Appendix J.

Also from Appendix J (Equation J-32), the joint probability density function in the time domain is given by

$$f(\Delta T_N, \Delta T_M) = \frac{1}{\sigma_x^2 \pi\sqrt{3}} \exp\left[-2/3 \frac{\Delta T_N^2 + \Delta T_M^2 - \Delta T_M \Delta T_N}{\sigma_x^2}\right] \tag{60}$$

For comparison in the space domain with the case of four observers, make the following conversion. If increments in the space domain are related to increments in the time domain by a constant, then

$$f(n, m)\, dA_s = f(\Delta t_n, \Delta t_m)\, dA_t$$

where dA_s and dA_t are the respective two-dimensional increments. The constants referred to are:

$$\Delta T_N = \frac{n}{\Gamma_n}, \qquad d\Delta T_N = \frac{dn}{\Gamma_n}$$

$$\Delta T_M = \frac{m}{\Gamma_m}, \qquad d\Delta T_M = \frac{dm}{\Gamma_m}$$

$$\sigma_x = \frac{\sigma_n}{\Gamma_n} = \frac{\sigma_m}{\Gamma_m}$$

Then

$$dA_s = dndm$$

$$dA_t = d\Delta T_N d\Delta T_M$$

and by substitution into Equation (60):

$$f(n,m)\,dA_s = \frac{\Gamma_n \Gamma_m}{\sigma_n \sigma_m \pi \sqrt{3}} \exp\left[-\frac{2}{3}\left(\frac{n^2}{\Gamma_n^2 \sigma_x^2} + \frac{m^2}{\Gamma_m^2 \sigma_x^2} - \frac{nm}{\Gamma_n \Gamma_m \sigma_x^2}\right)\right] \cdot \frac{dm}{\Gamma_m} \frac{dn}{\Gamma_n}$$

or

$$f(n,m) = \frac{1}{\sigma_n \sigma_m \pi\sqrt{3}} \exp\left[-\frac{2}{3}\left(\frac{n^2}{\sigma_n^2} + \frac{m^2}{\sigma_m^2} - \frac{nm}{\sigma_n \sigma_m}\right)\right] \tag{61}$$

is the probability density function in the space domain, comparable to equation (48) for the case of four observers. Further, in comparison, the results of Appendix D are as follows:

$$\alpha = \frac{1}{2\beta\sqrt{2\pi K}} \int_{-1}^{+1} [\Phi_1(x) - \Phi_2(x)] \exp\left[-\frac{x^2}{2K\beta^2}\right] dx \tag{62}$$

However,

$$\Phi_1(x) = \frac{\sin\theta_x}{\sqrt{3/2K}}\left[(x\cot\theta + \sqrt{1-x^2}) - \frac{x}{2\beta\sqrt{3/2K}}\right] \tag{63}$$

$$\Phi_2(x) = \frac{\sin\theta_x}{\sqrt{3/2K}}\left[(x\cot\theta - \sqrt{1-x^2}) - \frac{x}{2\beta\sqrt{3/2K}}\right] \tag{64}$$

and the symmetry prevails such that the conditions and values of Table I apply directly to this analysis.

Figure 6 presents the results of these equations to determine the circle R versus the required probability α .

Referring to the curves of both Figures 5 and 6, the abscissae are K and σ_s where σ_s represents σ_m if $\sigma_n/\sigma_m > 1$ and also represents σ_n if $\sigma_m/\sigma_n > 1$, provided either ratio is equal to the value of the β for the curves.

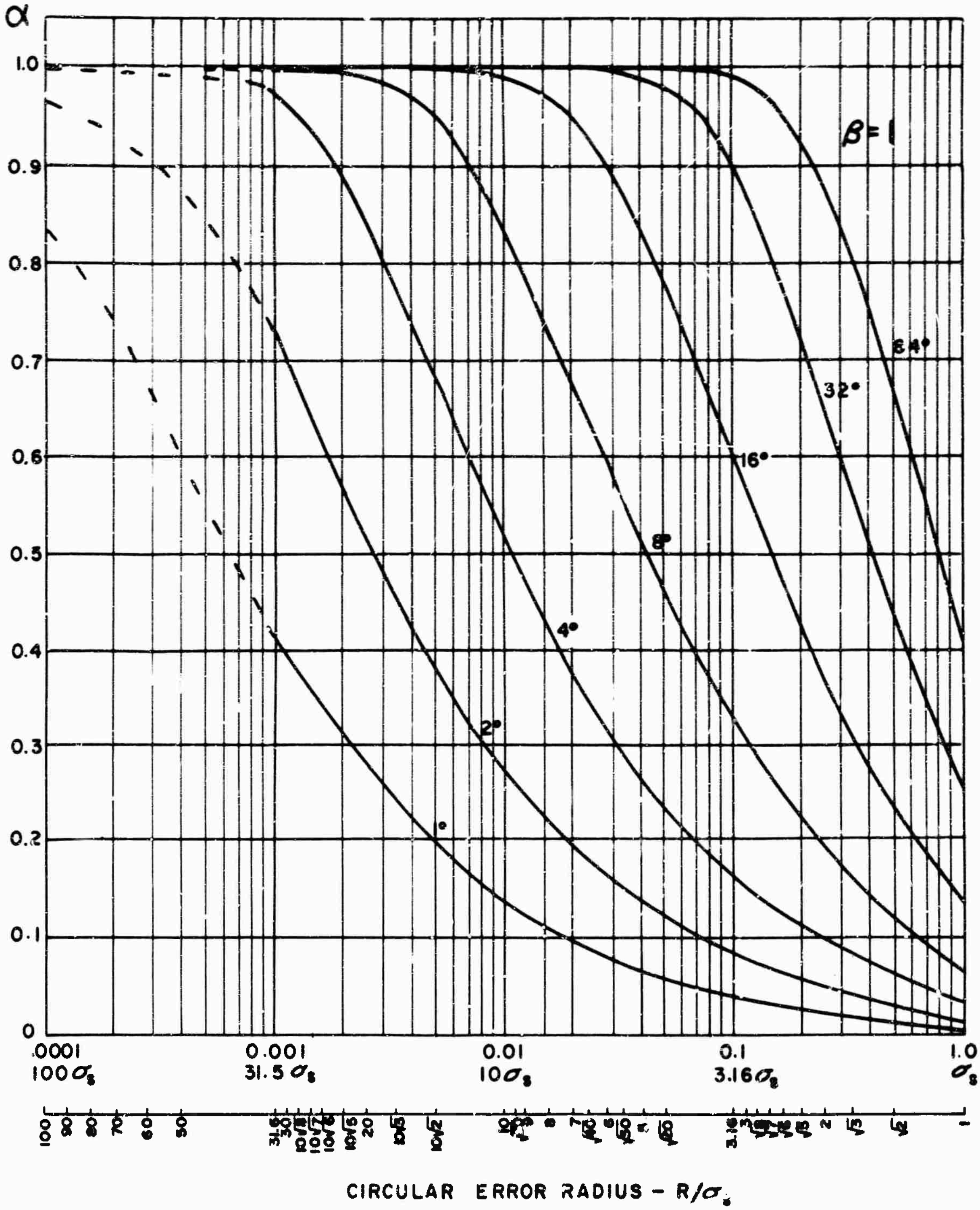


Figure 6(a). QCEP Radii Versus Probability α for the Case of Three Observers

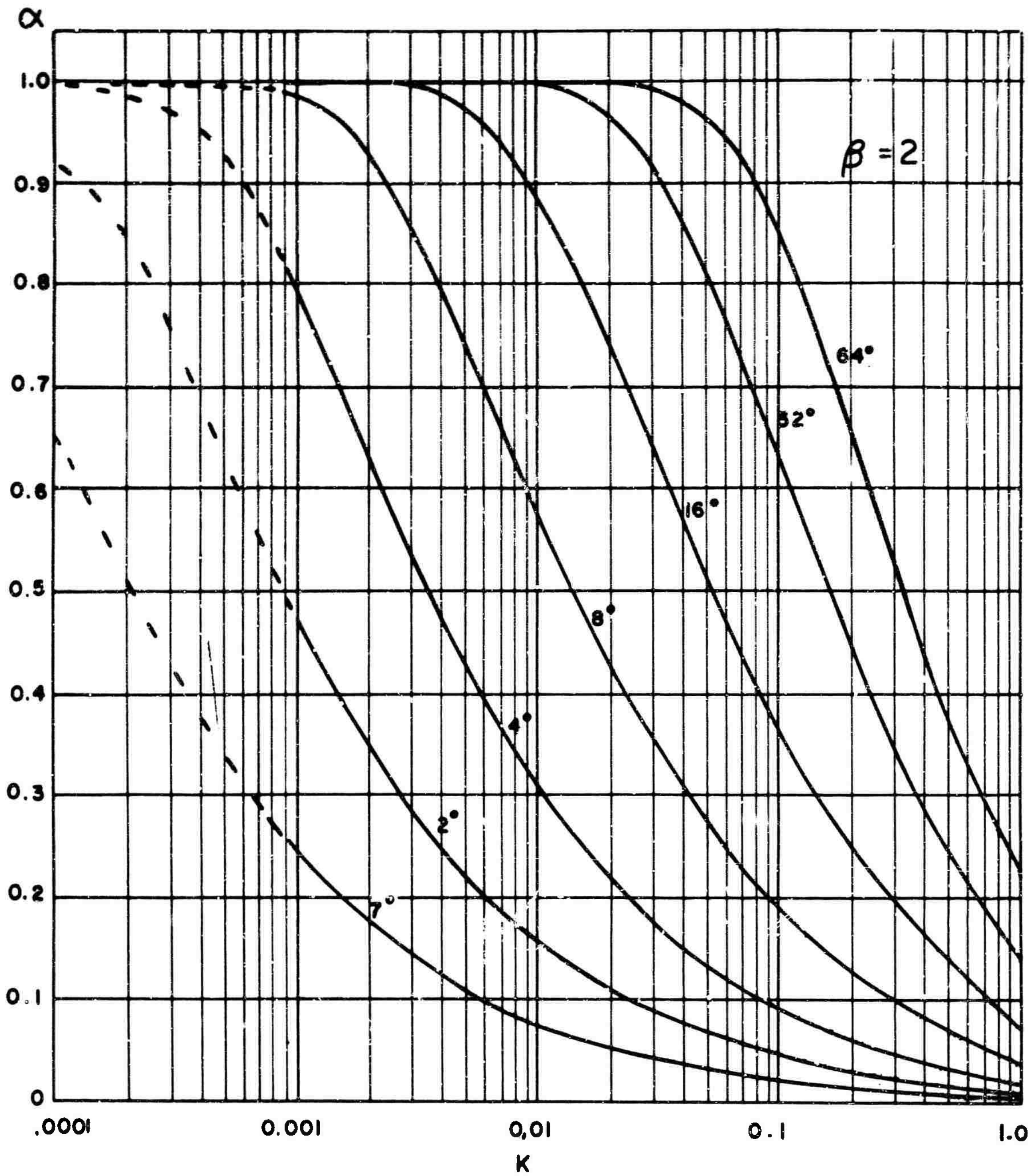


Figure 6(b). QCEP Radii Versus Probability α for the Case of Three Observers

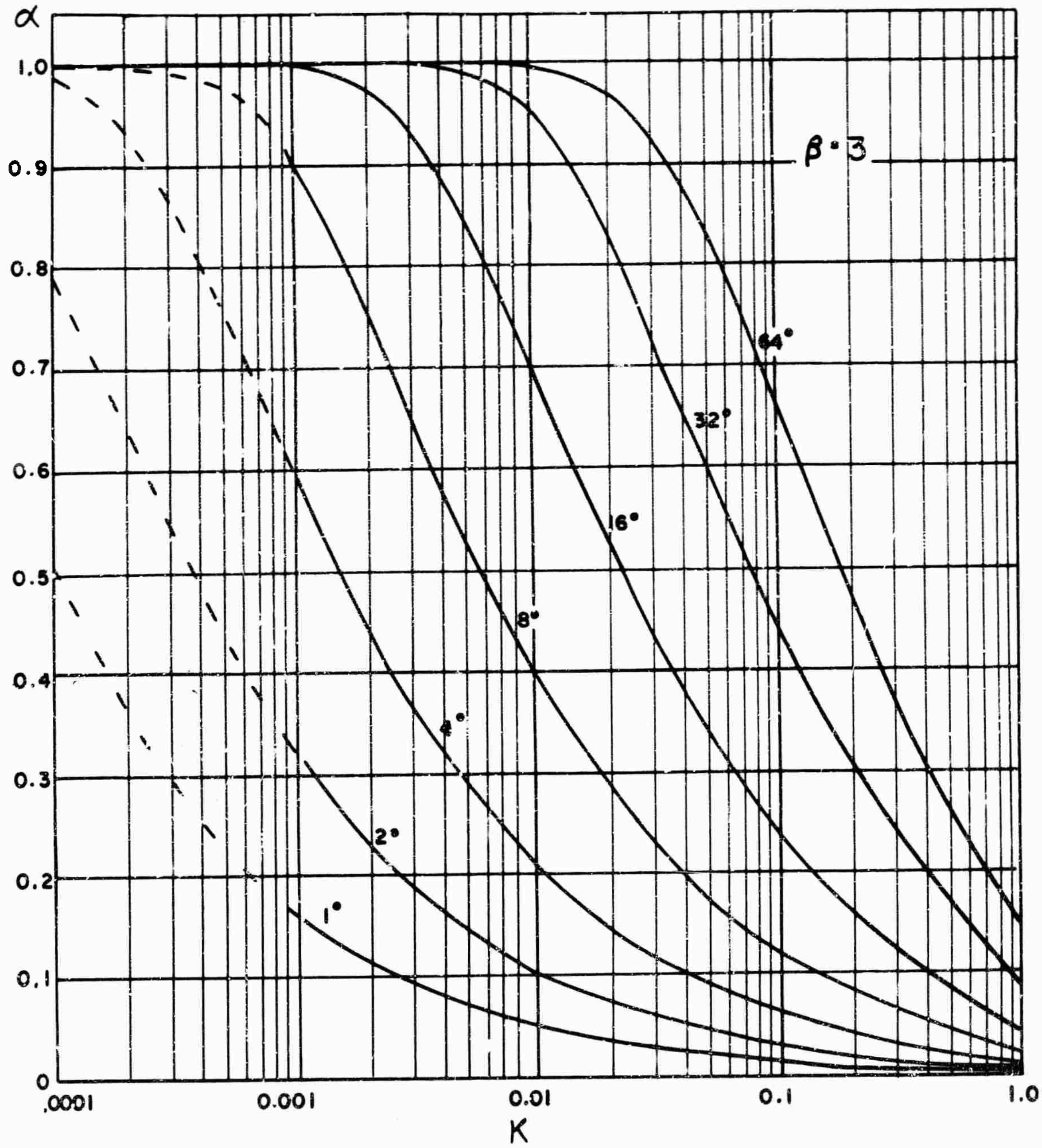


Figure 6(c). QCEP Radii Versus Probability α for the Case of Three Observers

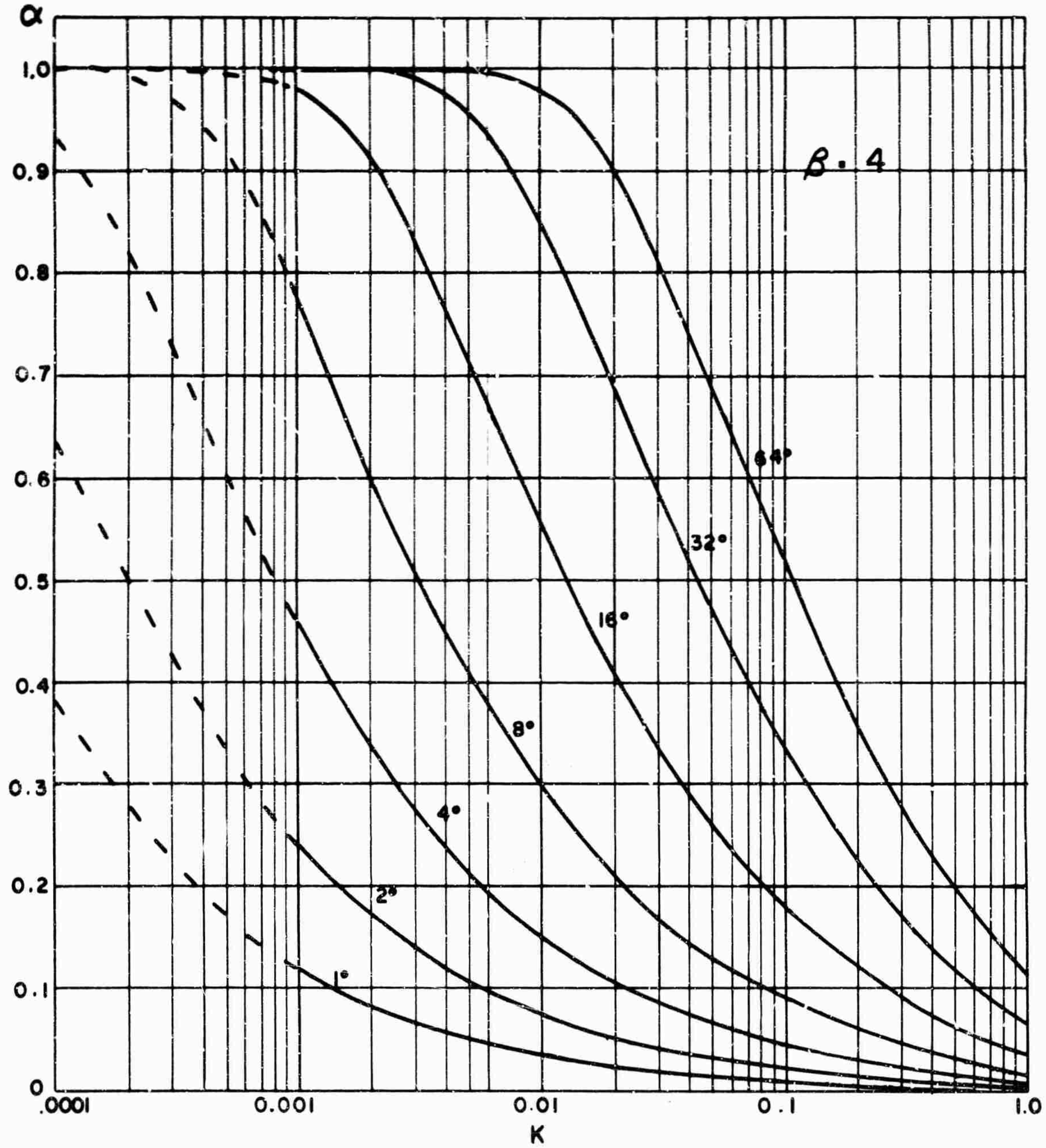


Figure 6(d). QCEP Radii Versus Probability α for the Case of Three Observers

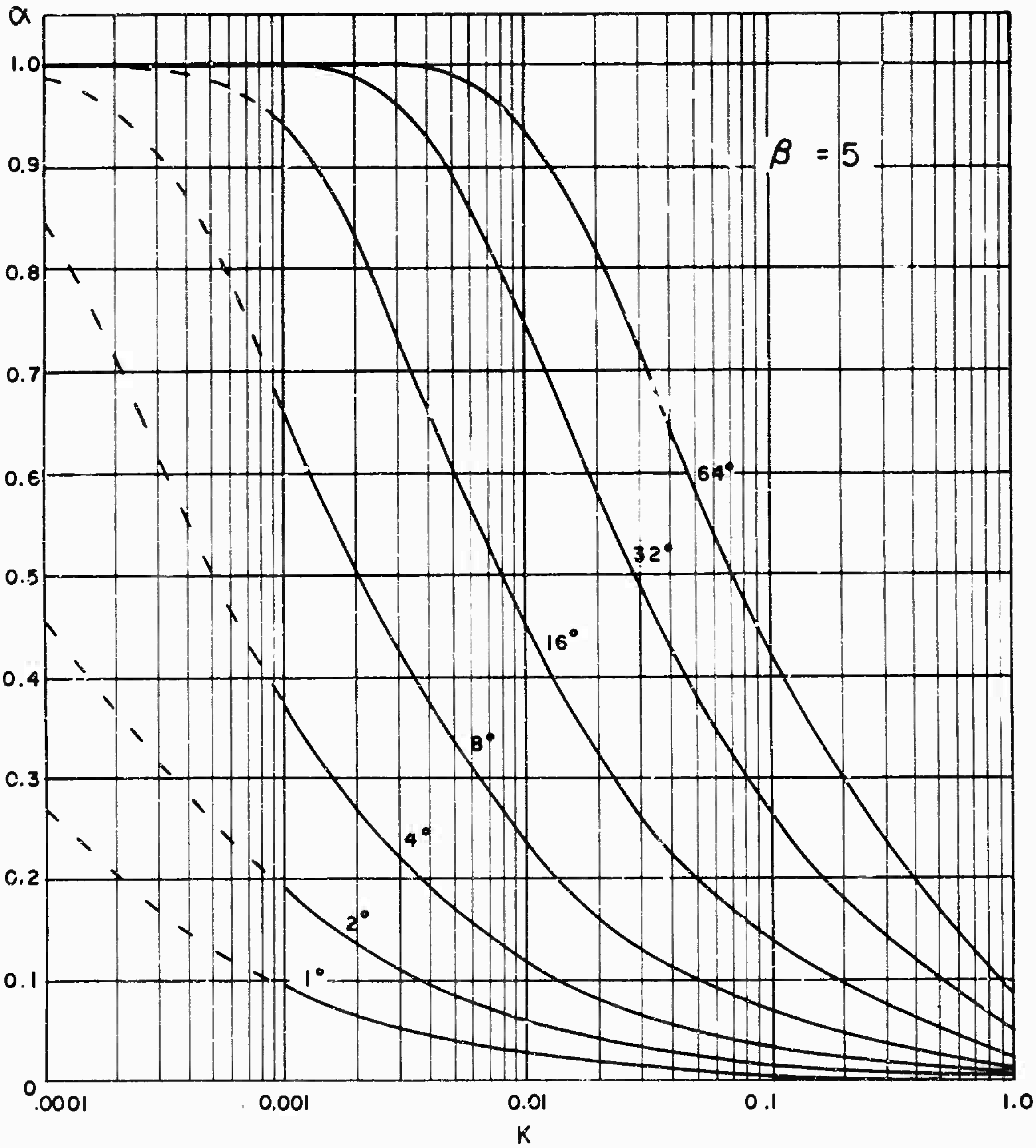


Figure 6(e). QCEP Radii Versus Probability α for the Case of Three Observers

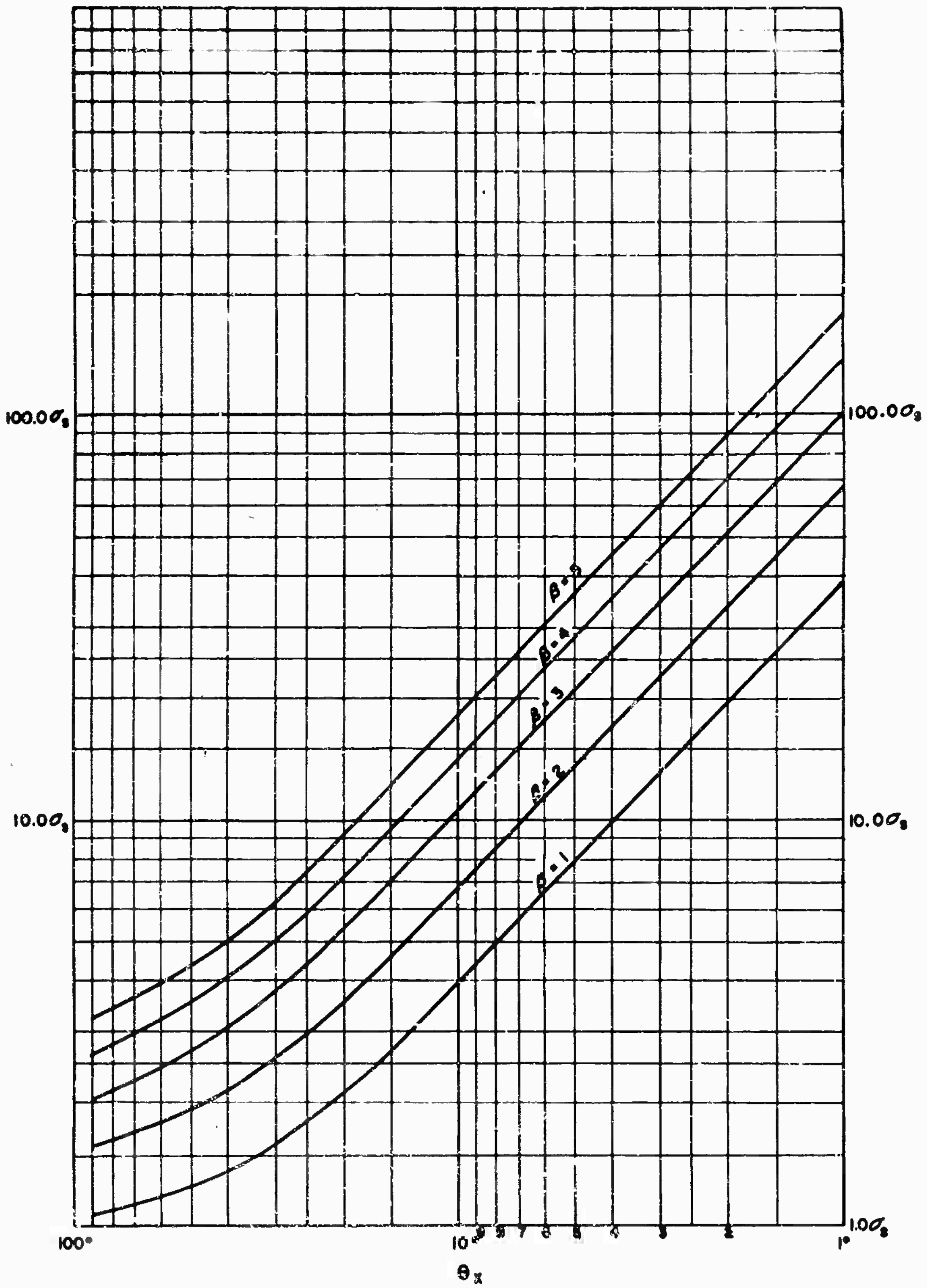


Figure 6(f). QCEP Radii Versus Probability α for the Case of Three Observers

VII. COMPARISON OF RESULTS OF FOUR OBSERVERS WITH THREE OBSERVERS

A comparison will now be made of three to four observers! If the corresponding β curves of probability for four observers are placed on top of the curves for three observers, there is a nearly constant shift of the former to the left or to greater radii. Table II gives a spectrum sampling and comparison of values throughout the systems of curves. A fairly constant ratio of R_4/R_3 is obtained for a given β. The averages decrease with increased β with some indication of approaching 1.0 as $\beta \rightarrow \infty$. No effort was made to deduce or prove this theory.

TABLE II

COMPARISON OF QCEP RADII FOR THREE AND FOUR OBSERVERS

		QCEP RADIUS		RATIO	
β	θ_x	Four Observers	Three Observers	R_4/R_3	Average
1	1°	52.	38.8	1.3	1.32
	2°	26.4	20.0	1.32	
	4°	13.4	9.75	1.37	
	8°	6.86	5.00	1.37	
	16°	3.56	2.65	1.34	
	32°	1.9	1.58	1.20	
2	1°	83.6	67.8	1.23	1.25
	2°	41.8	33.2	1.26	
	4°	21.4	16.7	1.28	
	8°	10.8	8.50	1.27	
	16°	5.48	4.36	1.26	
	32°	2.93	2.49	1.18	
3	1°	----	----	----	1.13
	2°	54.8	50.0	1.10	
	4°	29.2	25.3	1.15	
	8°	14.65	12.8	1.14	
	16°	7.22	6.48	1.12	
	32°	4.0	3.54	1.12	
4	1°	----	----	----	1.12
	2°	----	----	----	
	4°	39.1	35.4	1.10	
	8°	20.0	17.6	1.14	
	16°	10.0	8.85	1.13	
	32°	5.25	4.75	1.10	
5	1°	----	----	----	1.11
	2°	----	----	----	
	4°	----	----	----	
	8°	24.7	22.1	1.12	
	16°	12.4	11.2	1.10	
	32°	6.5	5.9	1.10	

VIII. COMPARISON OF QCEP's TO THE UNIFORM PROBABILITY OF ERROR

Another interesting comparison results with reference to Equations (16), (17), and (18). While these referred to a somewhat academic situation of uniform or equal probability of error magnitude, the following analysis indicates that these formulas may have more value than passing academic interest.

Since in Equations (16), (17), and (18) the value of ϵ is indeterminate, or at least somewhat arbitrary, a very feasible value to assign would be that derived from a timing error equal to the standard deviation of Gaussian timing errors. Thus, assuming

$$\epsilon = v\,\sigma_x \tag{65}$$

immediately forms a common bond between the two systems of measure. To continue this translation in Equation (16)

$$R = \frac{v\,\sigma_x}{C}\sqrt{\frac{r_n r_m}{2\pi \sin\theta_n \sin\theta_m \sin\theta_x}}$$

$$= \sqrt{\frac{2}{\pi \sin\theta_x} \times \frac{v\,r_m}{2C\sin\theta_n}\sigma_x \times \frac{v\,r_n}{2C\sin\theta_n}\sigma_x} \tag{66}$$

$$= \sqrt{\frac{2}{\pi \sin\theta_x}\Gamma_m \sigma_x \Gamma_n \sigma_x}$$

$$R = \sqrt{\frac{2}{\pi \sin\theta_x}\sigma_m \sigma_n} \tag{67}$$

Also, using

$$\sigma_m = \beta\,\sigma_n$$

$$R = \sigma_n\sqrt{\frac{2\beta}{\pi \sin\theta_x}}\;[R < \sigma_n < \sigma_m] \tag{68}$$

Similarly, translation of Equation (17) gives

$$R = \frac{\sigma_n}{2\sin\theta_x}\quad[\sigma_n < R < \sigma_m] \tag{69}$$

and for Equation (18)

$$R = \frac{\sigma_n}{2}\sqrt{\frac{\beta^2}{2\sin^2\theta_x} + 4} \quad [\sigma_n < \sigma_m < R] \tag{70}$$

These formulas are also adjusted to the condition, a requirement for consistency, that,

$$\sigma_m \geq \sigma_n$$

Doing this makes it possible to look up QCEP's from Figures 5 and 6 for the same θ_x and β as used in the above equations. This was done for the set of points θ_x, β given in Table III. Since reading the curves of Figure 5(f) and Figure 6(f) above 32° results in significant inaccuracies and since formulas (16), (17), and (18) incorporate approximations, the final tabulated results should be viewed with a grain of salt. However, looking at the ratio of QCEP's of three Gaussian observers to the hard limited equal probability error makers (R_{3G}/R_u) there is a strong indication that for rapid calculation the use of

$$R_{3G} = 1.3\ R_u$$

would give figures within 10% error. R_u would have to be calculated in accordance with Equations (68), (69), and (70). Another relationship resulting from the above substitutions is

$$\Delta S_n = 2\,\sigma_n$$

$$\Delta S_m = 2\,\sigma_m$$

The criteria for the use of Equations (68), (69), or (70) have been added in terms of σ_n and σ_m as shown.

TABLE III

COMPARISON OF QCEP RADII FOR GAUSSIAN AND UNIFORM ERROR DISTRIBUTION

			QCEP RADIUS (σ_n = 1 Mi.)			
		R_U	Gaussian R_G			RATIO
θ_x	β	Uniform	Four Observer	Three Observer	Equation	R_{3G}/R_U
2°	1	14.3 mi.	27.5	19.5	70	1.36
8°	1	3.7 mi.	6.8	5.0	70	1.35
40°	1	.99 mi.	1.6	1.38	68	1.38
64°	1	.84 mi.	1.15	1.14	68	1.36
2°	3	43.0 mi.	57.0	51.0	70	1.19
8°	3	10.7 mi.	15.4	13.0	70	1.22
40°	3	2.3 mi.	3.4	3.0	69	1.28
64°	3	1.67 mi.	2.5	2.32	69	1.40
2°	5	71.5 mi.	95.0	94.5	70	1.32
8°	5	18.1 mi.	24.5	22.3	70	1.23
40°	5	3.9 mi.	5.3	5.0	69	1.28
64°	5	2.78 mi.	4.0	3.76	69	1.35

IX. AN EXAMPLE OF QCEP RADIUS DETERMINATION

The following example is provided to tie some of these things together in an application. Assume an electromagnetic phenomenon as the system medium. Further assumptions:

1. The base lines = 200 miles = 2C
2. $\angle\theta = 130^o$ (Fig. 5)
3. $\angle e = 25^o$ (Fig. 5)
4. $\angle f = 25^o$ (Fig. 5)
5. A's clock reads 3.29459025 sec.
6. B's clock reads 3.2968855 sec.
7. C's clock reads 3.29337755 sec.

The first calculation is to obtain the line, d, connecting base line centers

$$d = C \frac{\sin \phi}{\sin e} = 100 \frac{\sin 50^o}{\sin 25^o} = 181.1 \text{ miles}$$

Next using Equations (5) and (6), the angles θ_n and ϑ_m are determined as follows:

$$t_A - t_B = 901.7 \text{ usec.}$$

$$t_B - t_C = 311.0 \text{ usec.}$$

$$v = \text{velocity of light} = 0.186 \times 10^6$$

$$\theta_n = \cos^{-1} \frac{v(t_A - t_B)}{2C} = \cos^{-1} \frac{0.186 \times 10^6 \times 902 \times 10^{-6}}{200}$$

$$\theta_n = \cos^{-1} 0.8386 = 33^o$$

$$\theta_m = \cos^{-1} \frac{v(t_B - t_C)}{2C} = \cos^{-1} \frac{0.186 \times 10^6 \times 311 \times 10^{-6}}{200}$$

$$\theta_m = \cos^{-1} 0.2588 = 75^o$$

Then, from Equation (8)

$$\theta_x = 180^o + 33^o - (130^o + 75^o)$$

$$\theta_x = 8^o$$

Equations (10) and (11) give

$$r_n = \frac{d \sin (\theta_m - e)}{\sin \theta_x}$$

$$r_n = \frac{181.1 \sin 50^o}{\sin 8^o} = 1000 \text{ miles}$$

$$r_m = \frac{d \sin (\theta_n + f)}{\sin \theta_x}$$

$$r_m = \frac{181.1 \sin 58^o}{\sin 8^o} = 1107 \text{ miles}$$

From Equations (24) and (25)

$$\Gamma_n = \frac{v\, r_n}{2C \sin \theta_n}$$

$$\Gamma_n = \frac{0.186 \times 10^6 \times 1000}{200 \sin 33^o} = 1.706 \times 10^6 \text{ miles per second}$$

$$\Gamma_m = \frac{v\, r_m}{2C \sin \theta_m}$$

$$\Gamma_m = \frac{0.186 \times 1107}{200 \sin} = 1.065 \times 10^6 \text{ miles per second}$$

As a final supposition, assume that statistical data available supports a figure of $\sigma = 0.707$ microsecond per observer,

or

$$\sigma_x = \sqrt{2}\, \sigma = 1 \text{ usec. per pair}$$

then

$$\sigma_n = \Gamma_n\, \sigma_x = 1.706 \text{ miles}$$

$$\sigma_m = \Gamma_m\, \sigma_x = 1.065 \text{ miles}$$

And the last system parameter β is

$$\beta = \frac{\Gamma_n}{\Gamma_m} = \frac{1.708}{1.065} = 1.603$$

If then we wish to determine the QCEP for three observers for $\alpha = 0.5$, reading from Figure 6(f) by interpolating between $\beta = 1$ and $\beta = 2$ we get

$$R = \sqrt{50}\,\sigma_g = 7.07\,\sigma_m = 7.07 \times 1.065$$

$$R = 7.53 \text{ miles}$$

as the radius of the circle which contains the true point of emission with a probability of 0.5.

To summarize, if we are given the fix position computed from a data set of time differences and can calculate the system standard deviations (σ_n, σ_m) and crossing angle θ_x, the formulas given heretofore or the set curves of Figures 5 and 6 can be utilized to obtain the size of the circle of error for any given probability of error. Conversely, if one is speculating about a particular circle, the probability of a true fix being within that circle can also be read from these curves. While sufficient data has not yet been analyzed to establish practical limits, it is believed that for

$$10^\circ < \theta_n \text{ and } \theta_m < 170^\circ$$

$$r_n \text{ and } r_m > 2C$$

This process will not result in errors greater than 5% in the calculation of errors.

X. DETERMINATION OF THE ELLIPTICAL ERROR PROBABILITY SURFACE FOR FOUR OBSERVERS

To continue with the analysis of error distribution, we will analyze the probabilities associated with areas bounded by an ellipse. Since, as shown in Appendix F, an ellipse has been found for which all points on the ellipse represent an equal probability density of fix error, this figure may be very important to systems applications. In Appendix F the complete development of the time domain to space domain transformation is given. As in the case of the QCEP, we will deal first with a four-observer system. The significant results are:

1. Given the circle which considers the probability of making timing errors of magnitude,

$$\Delta T_m^{\;2} + \Delta T_n^{\;2} = S^2 \tag{71}$$

which results in the spatial ellipse

$$\frac{1}{\sigma_x^{\;2}}\left(\frac{n^2}{2\,\Gamma_n^{\;2}} + \frac{m^2}{2\,\Gamma_m^{\;2}}\right) = 1 \tag{72}$$

where

$$\Delta T_n = \frac{n}{\Gamma_n}$$

$$\Delta T_m = \frac{m}{\Gamma_m}$$

2. The probability of a fix being anywhere on or within this ellipse is given by

$$\alpha = \frac{1}{2\pi\,\sigma_n\,\sigma_m} \int\int_A \exp\left[-\frac{1}{\lambda^2}\left(\frac{n^2}{2\sigma_n^{\;2}} + \frac{m^2}{2\,\sigma_m^{\;2}}\right)\right] dm\,dn$$

$$\alpha = 1 - e^{-\lambda^2} \tag{73}$$

where

$$\lambda = \frac{S}{\sqrt{2}\;\sigma_x} \tag{74}$$

For example, for the "natural ellipse"

$$\lambda = 1 , \quad S^2 = 2 \sigma_x^{\;2}$$

and

$$\alpha = 1-e^{-1} = 63.22\%$$

As a more general example, to find the elliptical constants for a given probability of error for the following case

$$\sigma_x = 1 \text{ usec}$$

$$C_B = 100 \text{ miles}$$

$$r_n = 1000 \text{ miles}$$

$$r_m = 1107 \text{ miles}$$

$$\theta_n = 33^{\circ}$$

$$\theta_m = 75^{\circ}$$

$$\alpha = 0.5$$

results in the same Γ_n, Γ_m as in the previous example, i.e.,

$$\Gamma_n = 1.706 \times 10^6$$

$$\Gamma_m = 1.065 \times 10^6$$

As shown in Appendix E, the general elliptical equation is given by

$$\frac{1}{\lambda^2} \left(\frac{n^2}{2 \sigma_x^{\;2} \Gamma_n^{\;2}} + \frac{m^2}{2 \sigma_x^{\;2} \Gamma_m^{\;2}} \right) = 1 \tag{75}$$

In terms of the major, minor axis concept, if

$$a^2 = 2 \lambda^2 \sigma_x^{\;2} \Gamma_n^{\;2}$$

$$b^2 = 2\lambda^2 \sigma_x^{\;2} \Gamma_m^{\;2}$$

$$\lambda = \sqrt{\log \frac{1}{1-\alpha}}$$

giving

$$\lambda = \sqrt{\log \frac{1}{1-0.5}} = 0.834$$

The ellipse would be

$$\frac{n^2}{a^2} + \frac{m^2}{b^2} = 1$$

with

$$a = 0.834 \times 1.414 \times 10^{-6} \times 1.706 \times 10^{6} = 2.012 \text{ miles}$$

$$b = 0.834 \times 1.414 \times 10^{-6} \times 1.065 \times 10^{6} = 1.256 \text{ miles}$$

and the ellipse becomes

$$\frac{n^2}{(2.012)^2} + \frac{m^2}{(1.256)^2} = 1$$

These are the parameters of a conformal ellipse drawn to orthogonal axes. To see the true spatial ellipse however, either a graphic projection onto the oblique axes will have to be performed or mathematical analysis using a, b, and θ_X to obtain an a', b' and axes shift angle ψ.

This section is devoted to a complete analysis and discussion of the true spatial ellipse as it exists on the true axes, m and n. The details of this analysis are given in Appendix H with significant results factored out here.

Figure 7 is presented as an example of a graphical analysis. To emphasize the elliptical appearance, it was decided to assume a

$$\beta = \frac{\Gamma_n}{\Gamma_m} \text{ of } 2$$

rather than carry through the previous numerical example. The assumed ellipse plotted on the orthogonal set n, z is

$$\frac{n^2}{16} + \frac{m^2}{4} = 1$$

Graphic transformation to the oblique axis set n, m is then performed as shown for the sample point, o. The ±z values of z are both used to show the two resulting transformed points, o'. The transformation is accomplished by projecting o onto z, then equilaterally from z onto m, thus making m equal to the z value, then perpendicularly down from m until it intersects the n value of o which is also a perpendicular drop from the n axis. While this solution was not performed with any great accuracy,

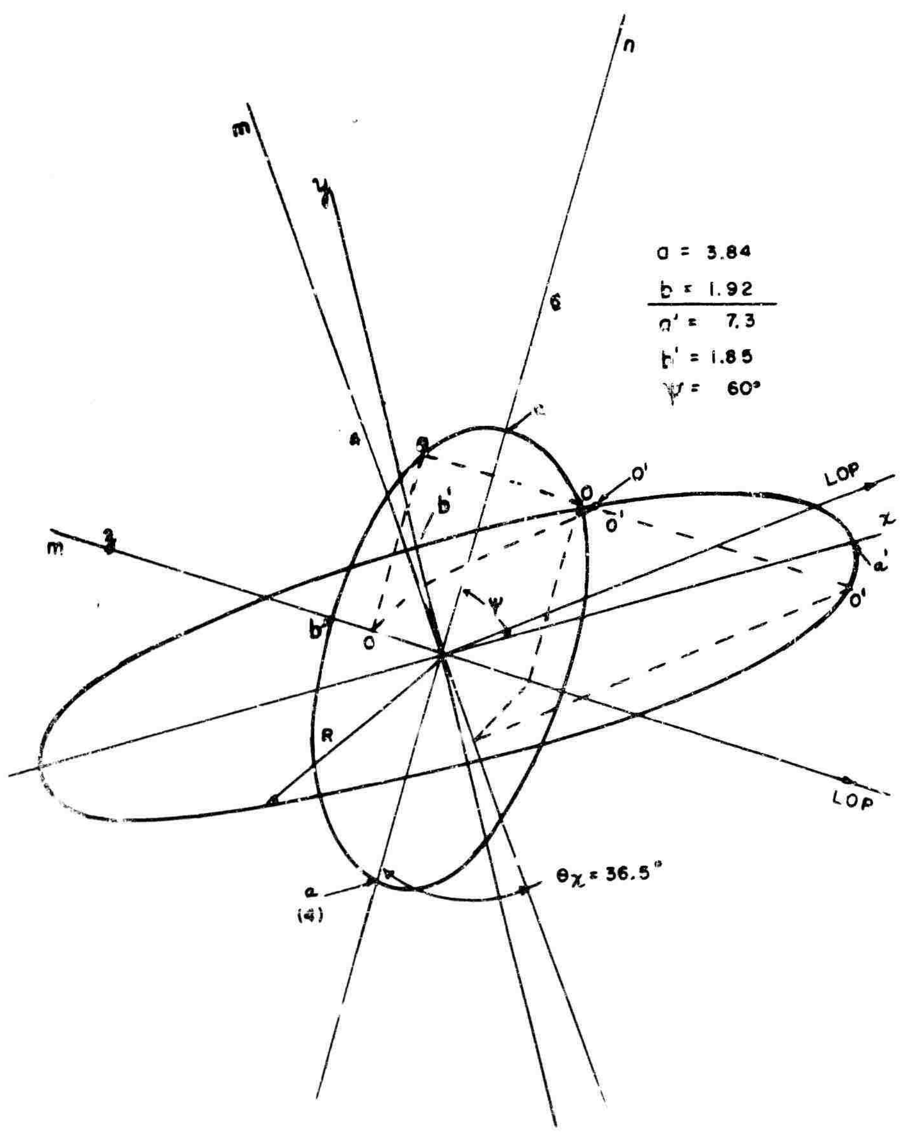


Figure 7. Transform of an Ellipse from Orthogonal to Oblique Axes

it is intended to display the significance and concept of the true ellipse on the oblique axes, and to provide approximate confirmation to the analytic equations. Measurements show the transformed major and minor axes to be approximately

$$a' = 7.3$$

$b' = 1.$[illegible]5

with an axis shift of approximately

$$\psi = 60^{\circ}$$

If we then let x and y represent the axes which contain a' and b', we have the quantitative relationship

$$\frac{x^2}{(7.2)^2} + \frac{y^2}{(1.85)^2} = \frac{x^2}{(a')^2} + \frac{y^2}{(b')^2} = 1$$

This then represents a complete description of a particular elliptical contour of constant probability. It can be made as accurate as the measurements of data (Δt, Γ) and graphical solution will allow.

The inconvenience, if not inaccuracy, of this method strongly indicated that a complete analytical solution would certainly be well worthwhile. The results of the analysis given in Appendix H are the following set of transforms from the orthogonal to the true oblique axes.

$$R^2_{Max/Min} = \frac{1}{\sin^2 \theta_x} \left\{ \frac{a^2}{2}\left(1 \pm \frac{x}{\sqrt{1+x^2}}\right) + \frac{b^2}{2}\left(1 \mp \frac{x}{\sqrt{1+x^2}}\right) \pm \frac{ab}{\sqrt{1+x^2}} \cos \theta_x \right\} \tag{76}$$

where

$$x = \frac{1 - b^2/a^2}{2b/a \cos \theta_x} = \frac{a/b - b/a}{2 \cos \theta_x}$$

$$a = \sqrt{2}\,\lambda\, \sigma_x \Gamma_n$$

$$b = \sqrt{2}\,\lambda\, \sigma_x \Gamma_m$$

Thus when $a > b$

$$R^2_{Max} = \frac{1}{2 \sin^2 \theta} \left\{ a^2 \left(1 + \sqrt{1 - \frac{1}{1+x^2}}\right) + b^2 \left(1 - \sqrt{1 - \frac{1}{1+x^2}}\right) + \frac{2ab \cos \theta}{\sqrt{1+x^2}} \right\} \tag{77}$$

$$R^2_{Min} = \frac{1}{2\sin^2\theta}\left\{a^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)-\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\} \tag{78}$$

and when $b > a$

$$R^2_{Max} = \frac{1}{2\sin^2\theta}\left\{a^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)+\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\} \tag{79}$$

$$R^2_{Min} = \frac{1}{2\sin^2\theta}\left\{a^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)-\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\} \tag{80}$$

Association of the quantities R_{Max}, R_{Min}, a', and b' with the axes x and y is somewhat arbitrary. In general with m and n oriented as shown in Figure 7, there will be an axis of transformation which lies in the quadrant defined by n and its orthogonal LOP. If we call this the x axis, then it will be the major axis regardless of whether a was longer or shorter than b, and the value of R_{max} would be applied to this axis; this x axis will always lie on or between the LOP's. Perhaps the previous formula would carry more general significance if it were worded

$$\frac{x^2}{R^2_{Max}} + \frac{y^2}{R^2_{Min}} = 1 \tag{81}$$

The angle of shift ψ of the pseudomajor axis to the true major axis is given by

$$\cos^2\psi = \frac{a^2\left(1+\frac{x}{\sqrt{1-\frac{1}{1+x^2}}}\right)}{\frac{1}{\sin^2\theta}\left[a^2\left(1+\frac{x}{\sqrt{1+x^2}}\right)+b^2\left(1-\frac{x}{\sqrt{1+x^2}}\right)+\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right]} \tag{82}$$

which can also be written (for $a > b$)

$$\cos^2\psi = \frac{\sin^2\theta\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)}{1+\sqrt{1-\frac{1}{1+x^2}}+\frac{b^2}{a^2}\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+\frac{2b/a\cos\theta}{\sqrt{1+x^2}}} \tag{83}$$

and for $b > a$

$$\cos^2 \psi = \frac{\sin^2 \theta \left(1+\sqrt{1 - \frac{1}{1+x^2}}\right)}{\frac{a^2}{b^2}\left(1 - \sqrt{1 - \frac{1}{1+x^2}}\right) + 1 + \sqrt{1 - \frac{1}{1+x^2}} + \frac{2\,{}^a/b \cos \theta}{\sqrt{1+x^2}}} \tag{84}$$

whereas in terms of the major axis transform

$$\cos^2 \psi = \frac{1 + \sqrt{1 - \frac{1}{1+x^2}}}{2 \text{ (Major Axis Xform)}^2} \tag{85}$$

Figure 8 presents a few of the curves for these functions. The curves are presented mostly to get a feel for the positioning of the axes of the true ellipse. The number of curves calculated and number of points per curve are not considered adequate in this report for good engineering accuracy. The equations are accurate, however.

As with most all special cases where certain results are invitingly obvious, it is interesting to note, as proven in Appendix H, that when $\Theta_x = 90°$, i.e., the true ellipse and the original ellipse are one and the same, solution of the above equation gives $\psi = 0°$. This result corroborates that no shift is necessary.

Also derived in Appendix H are transformation formulas for the major and minor axes. If R'_{Max} is considered the transformed, or major axis value, then R'_{Max} can be determined readily from

$$\left(\frac{R'_{Max}}{R_{Max}}\right)^2 = \frac{(R'_{Max})^2}{a^2} = \frac{1}{2 \sin^2 \theta} \left\{1 + \sqrt{1 - \frac{1}{1+x^2}} + \frac{b^2}{a^2}\left(1 - \sqrt{1 - \frac{1}{1+x^2}}\right) + \frac{2\,{}^b/a \cos \theta}{\sqrt{1+x^2}}\right\} \tag{77a}$$

where $a > b$. This is a major axis transformation formula. For $b > a$ we consider b and b' the major axes, and

$$\frac{(R'_{Max})^2}{b^2} = \frac{1}{2 \sin^2 \theta} \left\{\frac{a^2}{b^2}\left(1 - \sqrt{1 - \frac{1}{1+x^2}}\right) + \left(1+\sqrt{1 - \frac{1}{1+x^2}}\right) + \frac{2\,{}^a/b \cos \theta}{\sqrt{1+x^2}}\right\} \tag{79a}$$

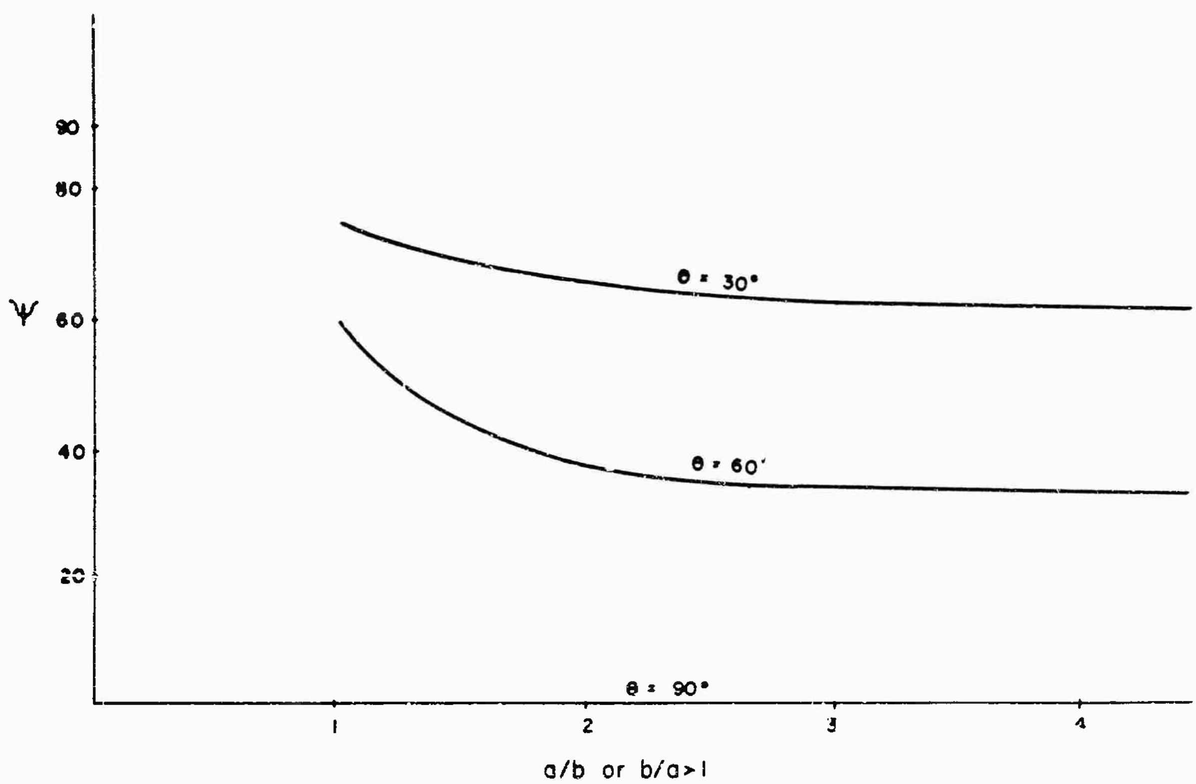


Figure 8. Major Axis Shift (Four Observers)

Similarly, the minor axis transformation is given by $a > b$

$$\frac{(R'_{Min})^2}{b^2} = \frac{1}{2\sin^2\theta}\left\{\frac{a^2}{b^2}\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)-\frac{2\,{}^a/_b\cos\theta}{\sqrt{1+x^2}}\right\} \quad (78a)$$

or, for $b > a$

$$\frac{(R'_{Min})^2}{a^2} = \frac{1}{2\sin^2\theta}\left\{1+\sqrt{1-\frac{1}{1+x^2}}+\frac{b^2}{a^2}\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)-\frac{2\,{}^b/_a\cos\theta}{\sqrt{1+x^2}}\right\} \quad (80a)$$

Figures 9, 10 present some of these curves of major and minor axes transformation. The transformed minor axis values approach 1 as a limit, i.e., the ellipse never gets any wider than its original value. The transformed major axis values, however, approach (a or b)/sin θ_x as their limit, meaning that as the crossing angle vanishes, the true spatial error ellipse becomes infinitely elongated. The choice of abscissa value was purely arbitrary, and whichever ratio was greater than one was selected. It is noted that for a/b, or β, greater than 5, the transforms are virtually at their limits. In fact, (a or b)/sin θ_x is a good approximation for $\beta > 2$.

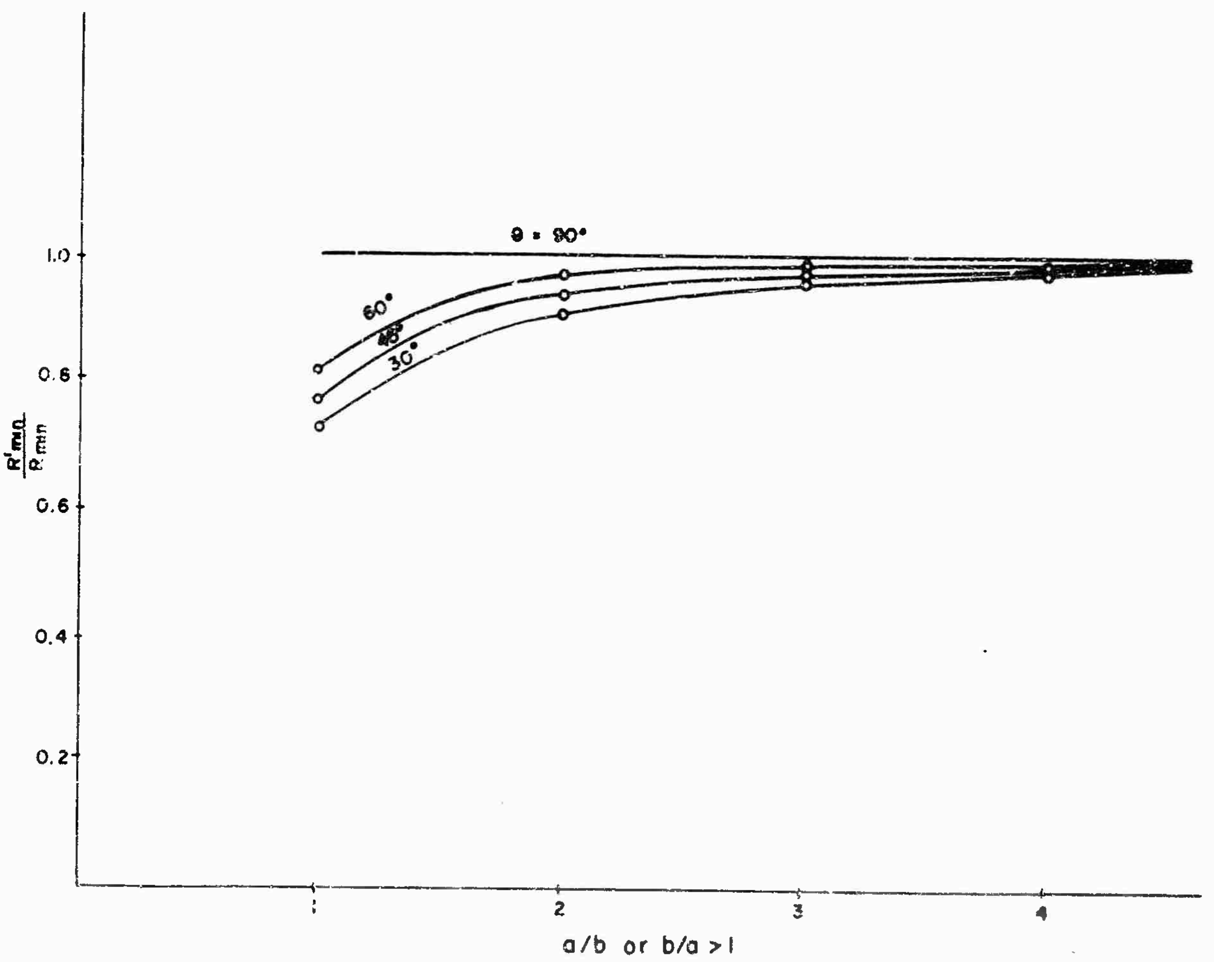


Figure 9. Minor Axis Transforms (Four Observers)

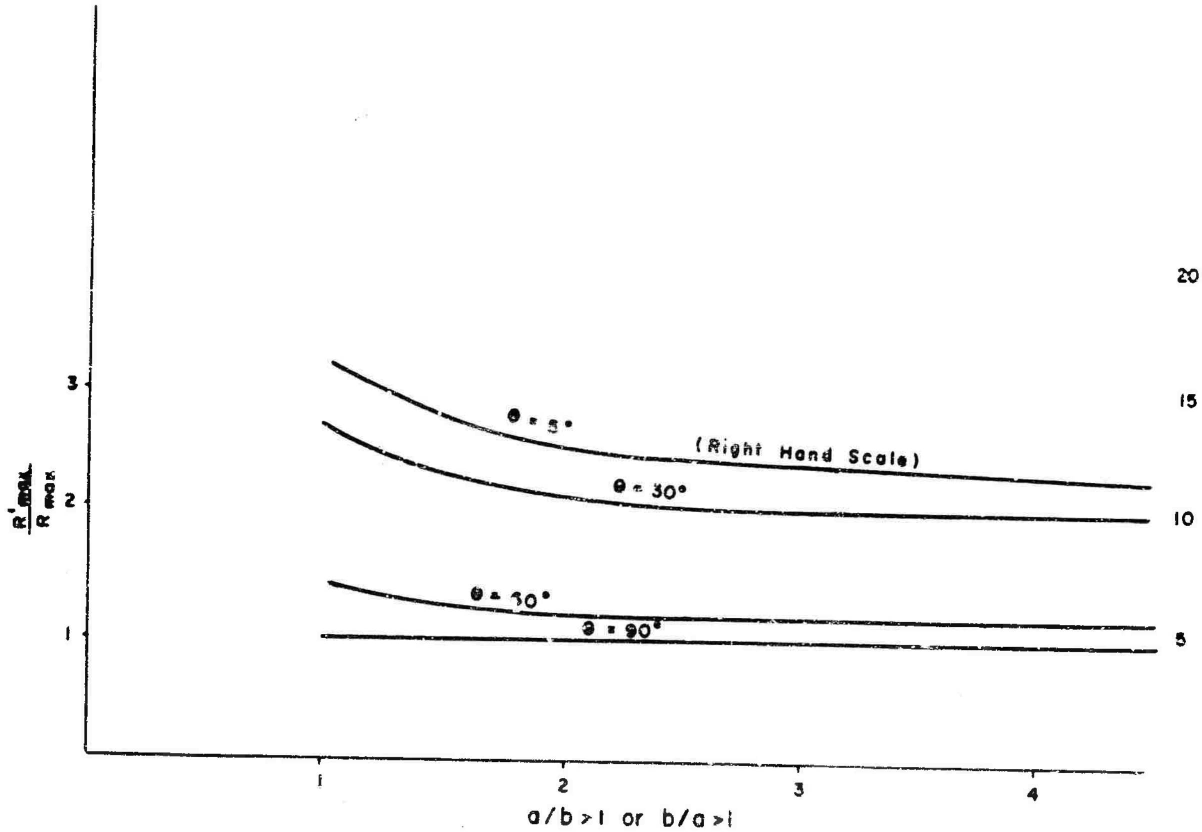


Figure 10. Major Axis Transforms (Four Observers)

XI. DETERMINATION OF THE EEP SURFACE FOR THREE OBSERVERS

Derivation of the true ellipse transforms for three observers appears in Appendix I, with the following results. The general formula for the major and minor axis is

$$R^2 = \frac{1}{\sin^2\theta_x}\left\{\frac{2}{3}\,b^2\left(1 \pm \sqrt{1 - \frac{1}{1+\xi^2}}\right) + \frac{2}{3}\,a^2\left(\pm\sqrt{1 - \frac{1}{1+\zeta^2}}\right) \pm \frac{4}{3}\,ab\sqrt{\left(1\pm\sqrt{1 - \frac{1}{1\mp\xi^2}}\right)\left(1\pm\sqrt{1 - \frac{1}{1+\zeta^2}}\right)}\cos\theta_x\right\} \tag{86}$$

and the transforms are obtained from

$$\frac{R}{a} = \frac{1}{\sin\theta}\sqrt{\frac{2}{3}\frac{b^2}{a^2}\left(1\pm\sqrt{1 - \frac{1}{1+\xi^2}}\right) + \frac{2}{3}\left(1\pm\sqrt{1 - \frac{1}{1+\zeta^2}}\right) \pm \frac{4}{3}\frac{b}{a}\sqrt{\left(1\pm\sqrt{1 - \frac{1}{1+\xi^2}}\right)\left(1\pm\sqrt{1 - \frac{1}{1+\zeta^2}}\right)}\cos\theta} \tag{87}$$

and

$$\frac{R}{b} = \frac{1}{\sin\theta}\sqrt{\frac{2}{3}\left(1\pm\sqrt{1 - \frac{1}{1+\xi^2}}\right) + \frac{2}{3}\frac{a^2}{b^2}\left(1\pm\sqrt{1 - \frac{1}{1+\zeta^2}}\right) \pm \frac{4}{3}\frac{a}{b}\sqrt{\left(1\pm\sqrt{1 - \frac{1}{1+\xi^2}}\right)\left(1\pm\sqrt{1 - \frac{1}{1+\zeta^2}}\right)}\cos\theta} \tag{88}$$

depending on the relative sizes of a and b in accordance with the following table:

FORMULA for

If	XFMax is	XFMin is
$a > b$	R/a (87)	R/b (88)
$b > a$	R/b (88)	R/a (87)

ξ and ζ are given by

$$\xi^2 = \frac{\left(2 - 2\frac{a}{b}\cos\theta_x - \frac{a^2}{b^2}\right)^2}{3\left(a^2/b^2 - 2\frac{a}{b}\cos\theta_x\right)^2}$$

$$\zeta^2 = \frac{\left(2 - 2^b/a\cos\theta_x - b^2/a^2\right)^2}{3\left(b^2/a^2 - 2^b/a\cos\theta_x\right)^2}$$

The rotation of the major axis, from its pseudo to true position is given by

$$\cos\psi = \frac{\sqrt{2/3\left(1\pm\sqrt{1-\frac{1}{1+\xi^2}}\right)}}{XF\ Max} \qquad [b > a] \tag{89}$$

$$\cos\psi = \frac{\sqrt{2/3\left(1\pm\sqrt{1-\frac{1}{1+\zeta^2}}\right)}}{XF\ Max} \qquad [a > b] \tag{90}$$

where in Equation (89) ψ is measured from the m axis and in Equation (90) ψ is measured from the n axis.

Contrary to the case of four observers where the resolved major axis always lies between the LOP's, the three observers case results in a resolved major axis which will, for the most part, lie outside the region defined by the LOP's. The only exception to this is when

$$1 < \beta < 2\cos\theta_x$$

and

$$\theta_x < 60^\circ$$

This suggests, depending on the cost of a fifth observer, that a five-observer system might produce an intersection of ellipses such that the area of uncertainty is significantly reduced.

Determination of the values of a and b requires a derivation similar to the case of four observers. In Appendix G the relationship between the size of the (concentric) ellipse and the probability of containment is given by

$$\alpha = 1-e^{-S^2} \tag{91}$$

Since this formula and Equation (73) are numerically the same for given values of λ or S, Figure 11 is presented as a quick reference of related values.

The CDC 1604B computer was again utilized to obtain the sample curves of Figures 12 and 13 showing the transform values as a function of a/b or β. Comparison of these figures with Figure I-2 shows that for $\theta < 60^\circ$ the maxima of the minor axis transforms and the minima of the major axis transforms coincide with the maxima for the curves of Figure I-2. Further, the contours of the minor axis transforms follow the contours of the quantity

$$\sqrt{1-\frac{1}{1+\xi^2}} \quad \text{or} \quad \sqrt{1-\frac{1}{1+\zeta^2}}$$

whichever contains the quantity a/b or b/a greater than unity.

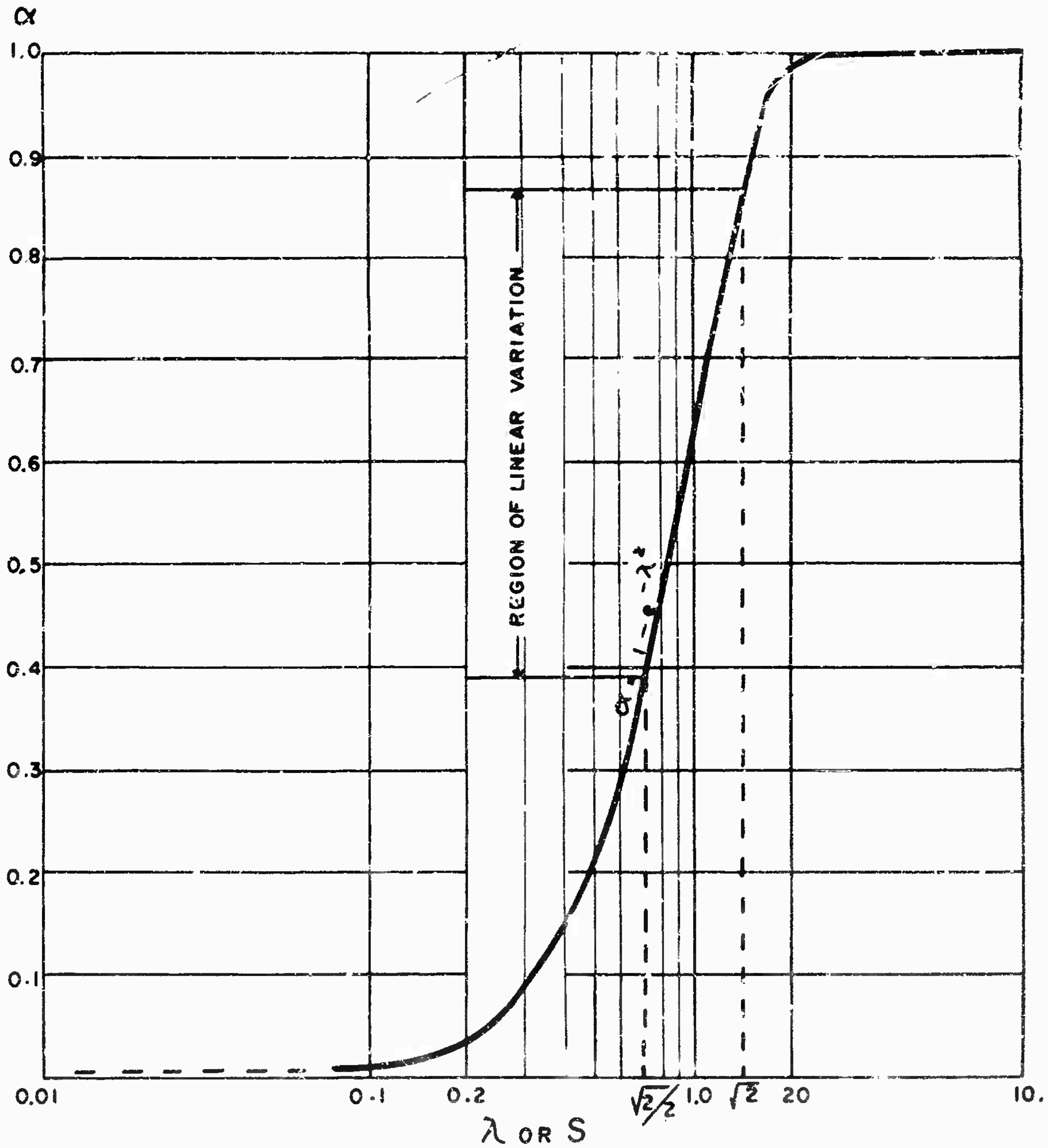


Figure 11. Probability vs. Concentric Ratio Factor of Ellipses

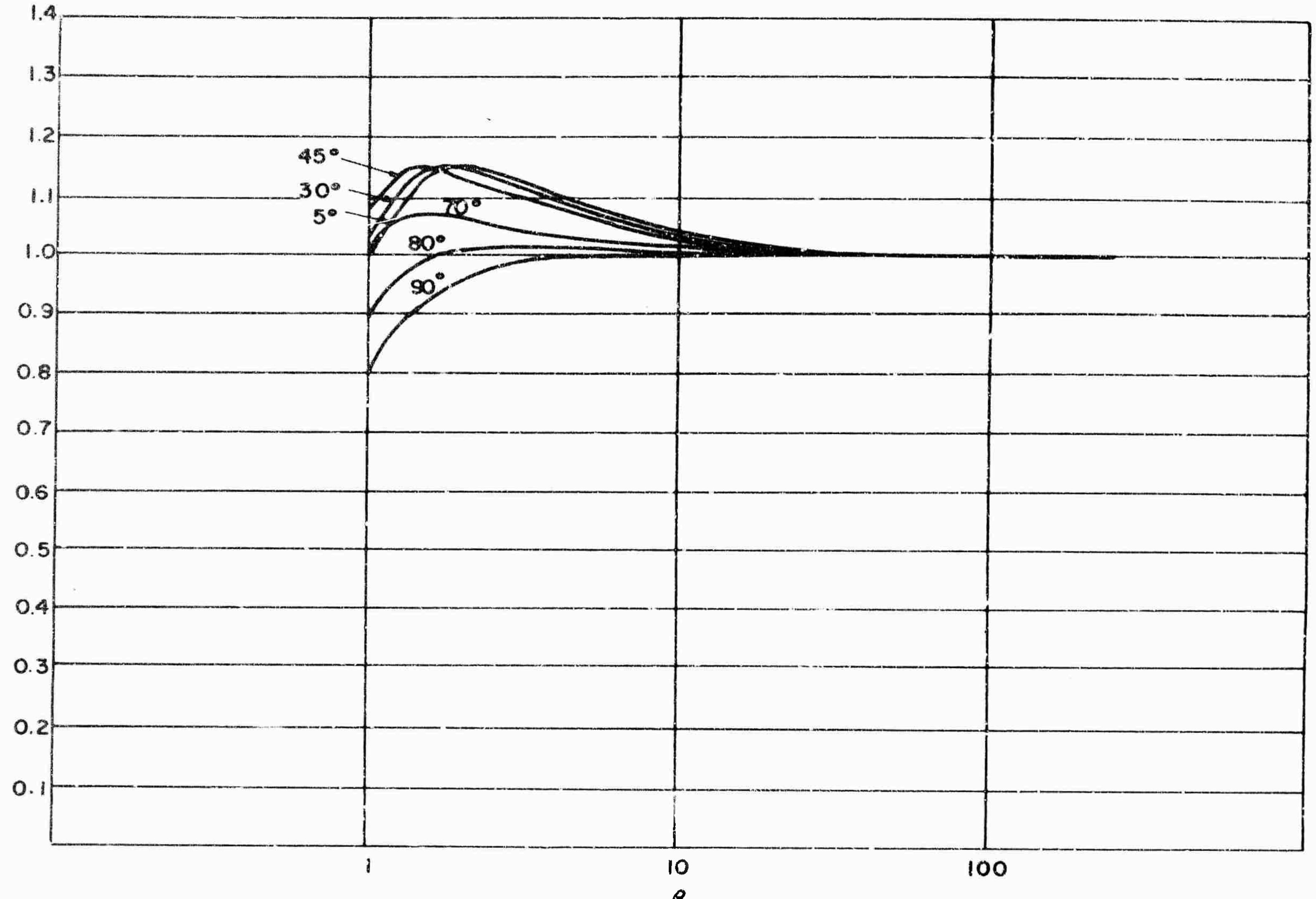


Figure 12. Minor Axis Transforms (Three Observers)

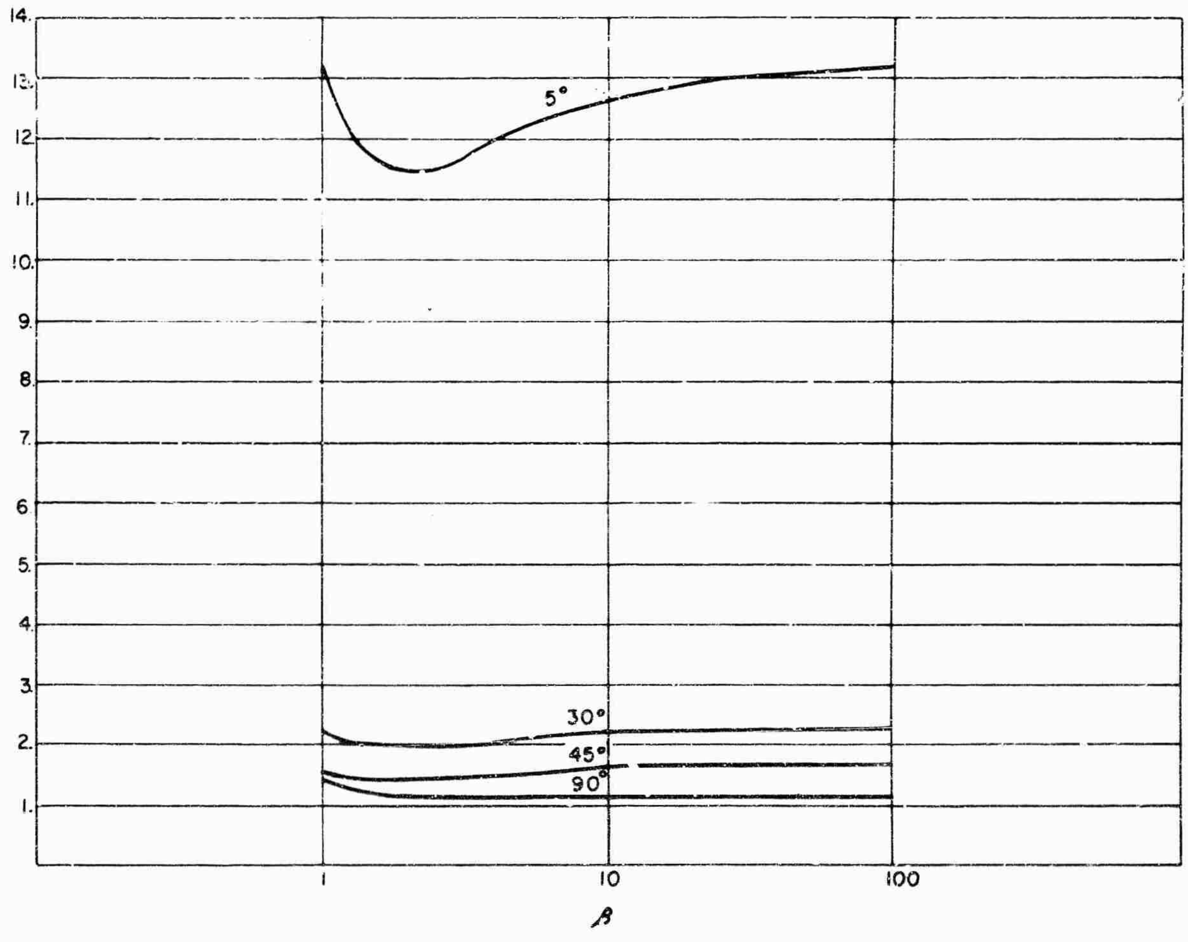


Figure 13. Major Axis Transforms (Three Observers)

As in the case of four observers, the curves in Figures 12 and 13 are not sufficient to provide general engineering calculations, although they are reasonably accurate for the values of θ_x chosen.

XII. COMPARISON OF EEP SURFACES FOR FOUR VERSUS THREE OBSERVERS

The subject comparison is not quite as straightforward as that of QCEP. To equate the two systems, the following approach was taken. Given the acceptable uncertainty or probability α we have

$$\lambda = S = \sqrt{\log \frac{1}{1-\alpha}} \tag{92}$$

This determines the time- or space-limited configuration

$$\left.\frac{\Delta T_m^{\ 2} + \Delta T_n^{\ 2}}{2\,\sigma_x^{\ 2}}\right|_{4\ \mathrm{OBS}} = \left.\frac{\Delta T_M^{\ 2} + \Delta T_N^{\ 2} - \Delta T_M\,\Delta T_N}{3/2\ \sigma_x^{\ 2}}\right|_{3\ \mathrm{OBS}}$$

or,

$$\left.\frac{m^2}{2\,\sigma_m^{\ 2}} + \frac{n^2}{2\,\sigma_n^{\ 2}}\right|_{4\ \mathrm{OBS}} = \left.\frac{M^2}{3/2\ \sigma_m^{\ 2}} + \frac{N^2}{3/2\ \sigma_n^{\ 2}} - \frac{MN}{3/2\ \sigma_m\,\sigma_n}\right|_{3\ \mathrm{OBS}} = S^2 = \lambda^2 \tag{93}$$

where upper case (M, N) is used to distinguish values from m, n and which in terms of the nontransformed ellipses is,

$$\left.\frac{m^2}{b_4^{\ 2}} + \frac{n^2}{a_4^{\ 2}}\right|_4 = \left.\frac{M^2}{b_3^{\ 2}} + \frac{N^2}{a_3^{\ 2}} - \frac{MN}{b_3\,a_3}\right|_3 = 1 \tag{94}$$

where

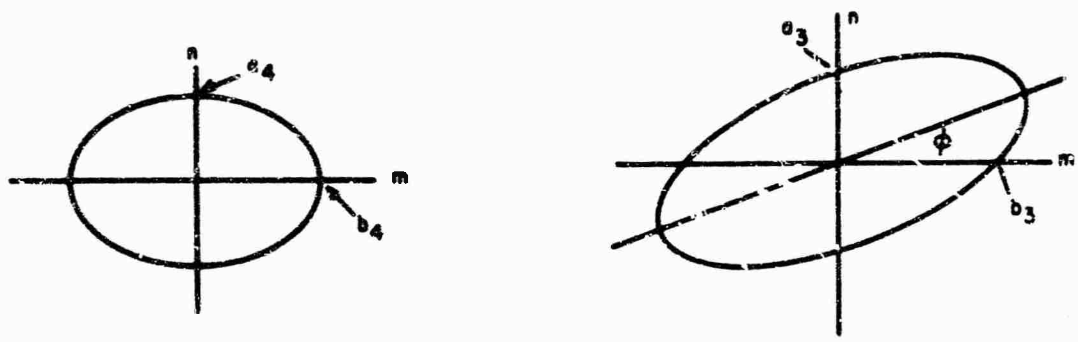
$$a_4 = \sqrt{2}\,\lambda\,\sigma_{n4}$$

$$b_4 = \sqrt{2}\,\lambda\,\sigma_{m4}$$

$$a_3 = \sqrt{3/2}\,S\,\sigma_{n3}$$

$$b_3 = \sqrt{3/2}\,S\,\sigma_{m3}$$

To recall the geometric significance of this we have



Therefore, if, as in the case of the QCEP, it is assumed that the systems are such that on an average or median, or any other basis

$$\frac{\sigma_{m4}}{\sigma_{n4}} = \frac{\Gamma_{m4}}{\Gamma_{n4}} = \beta_4 = \frac{\sigma_{m3}}{\sigma_{n3}} = \frac{\Gamma_{m3}}{\Gamma_{n3}} = \beta_3$$

Then

$$\frac{a_4}{a_3} = \frac{b_4}{b_3} = \frac{\sqrt{2}}{\sqrt{3/2}} = 1.15 \tag{95}$$

This advantage for three observers is not the final conclusion since the final transformed axis values are the conclusive factors. To derive equal β's the ratio desired is

$$\frac{\text{Major Axis (3 OBS)}}{\text{Major Axis (4 OBS)}} = \frac{1}{1.15} \; \frac{\text{XF Max}_3}{\text{XF Max}_4} \tag{96}$$

$$\frac{\text{Minor Axis (3 OBS)}}{\text{Minor Axis (4 OBS)}} = \frac{1}{1.15} \; \frac{\text{XF Min}_3}{\text{XF Min}_4} \tag{97}$$

Figure 14 gives the results of such a ratio comparison for two values of θ_x (30° and 90°). It can be shown that these curves do a flip-flop at the critical angle of $\theta_x = 60°$. It is again claimed that in systems applications the greatest concern is for $\theta_x < 60°$ as covering the majority of practical cases. Further, it is the elongation of the major axis which is the worst offender of, or produces the most damaging effect in, finding something or somebody. Thus, speaking of curves A and B, while the ellipse for three observers is a little fatter than for four, the major axis shows a considerable improvement for three observers.

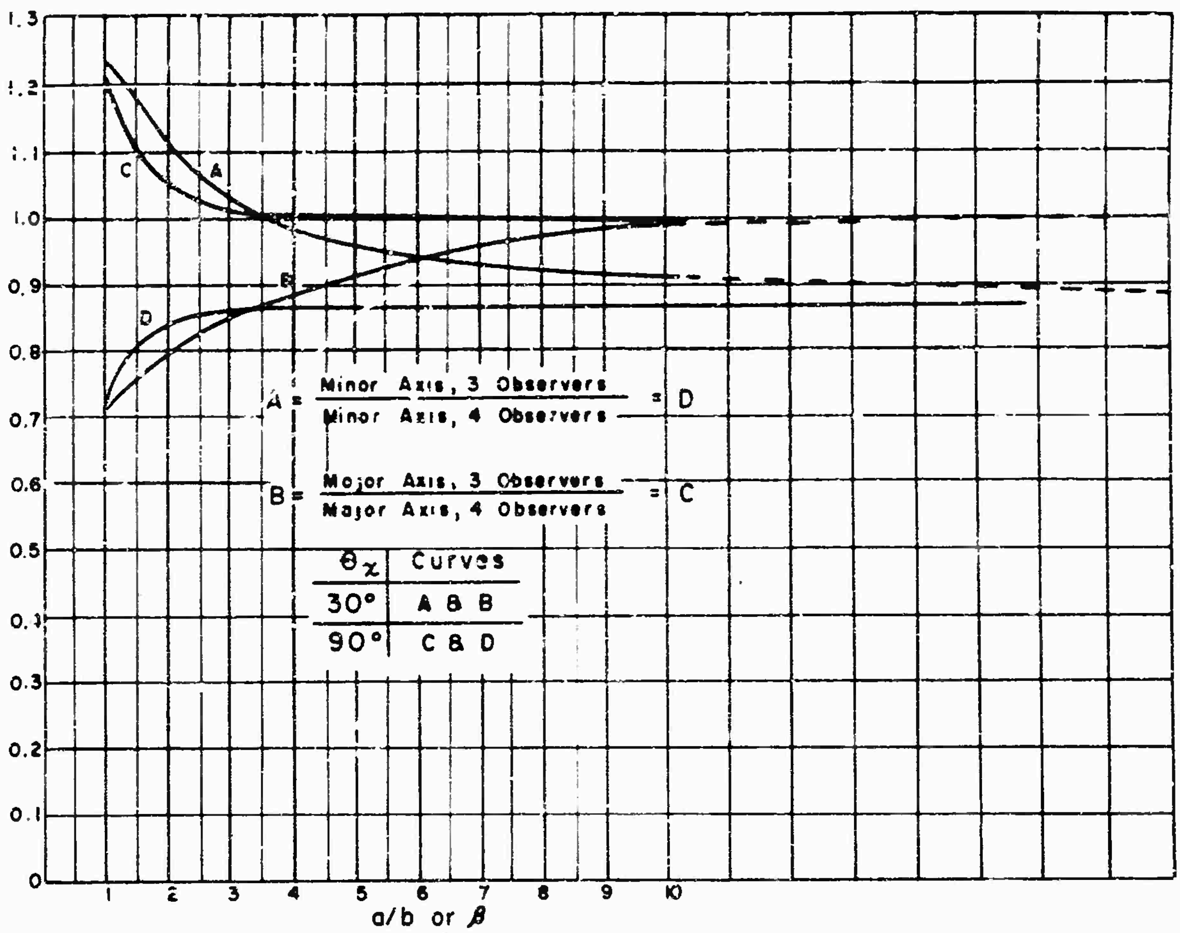


Figure 14. Comparison of Ellipses for Three and Four Observers

Returning to the example used for QCEP,

$$\sigma_n = 1.706 \text{ miles}$$

$$\sigma_m = 1.065 \text{ miles}$$

$$\beta = 1.603$$

$$\theta_x = 8^o$$

and for three observers

$$R_c = 7.53 \text{ miles}$$

We now wish to determine the ellipse for $\alpha = 0.5$ in order that direct comparisons can be made with QCEP. From Equation (22)

$$S = \sqrt{\log \frac{1}{1-\alpha}} = 0.834$$

and

$$a = \sqrt{3/2}\, S\, \sigma_n = \sqrt{3/2}\, 0.834 \times 1.706 = 1.74 \text{ miles}$$

$$b = \sqrt{3/2}\, S\, \sigma_m = \sqrt{3/2}\, 0.834 \times 1.065 = 1.09 \text{ miles}$$

Then for the major axis of the true ellipse

$$R_{max} = a\, XF_{max} = 1.74 \times 7.2$$

and for the minor axis

$$R_{min} = b\, XF_{min} = 1.09 \times 1.14$$

giving

$$R_{max} = 12.5 \text{ miles}$$

$$R_{min} = 1.24 \text{ miles}$$

For which, as discussed below

$$1.24 < 7.53 < 12.5$$

or

$$R_{min} < R_c < R_{max}$$

Thus we have a complete engineering description of the true spatial elliptical contour of a constant error probability. The position of the point P in question with respect to the system base line gives rise to the σ_n, σ_m and θ_x of the system, and the desired reliability of measurements, α, gives rise to the pseudo-elliptical quantities a and b. From these, the true ellipse of the probability α, the position of its major and minor axes, the values of these axes, and, if we wish, a complete sketch or plot of any desired ellipse can be described.

There is a somewhat crude yet interesting link between the QCEP and EEP which may be valuable for making system estimates. Observing Figure 15, it is apparent that there is a circle of radius R and an ellipse of major axis R_{max} and minor axis R_{min}, which by virtue of their respective areas of consideration will produce the same probability of fix.

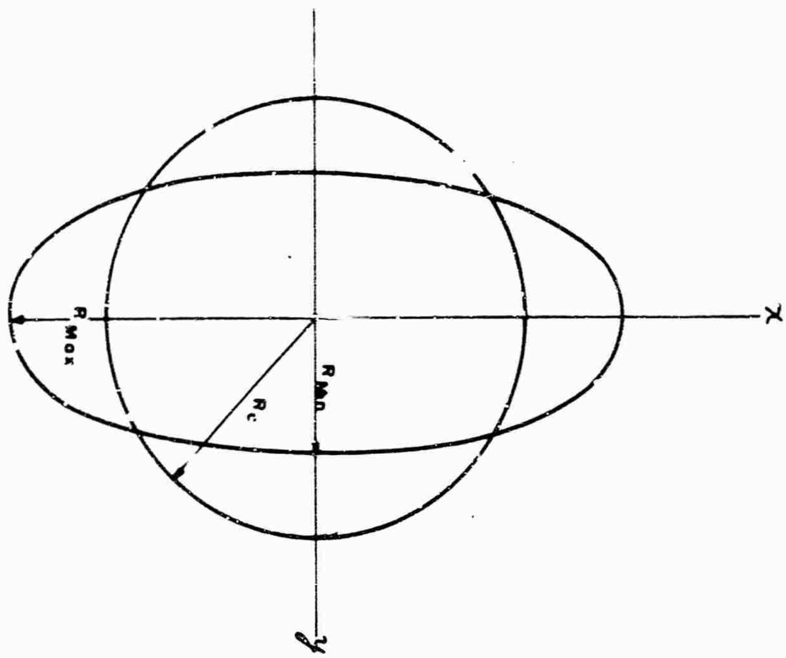


Figure 15. An EEP/QCEP Linkage

Thus we have the linkage

$$R_{min} < R_c < R_{max} \tag{98}$$

Since it is believed that the elliptical constants are more meaningful and more readily calculated than the circle, if the EEP is known, the circle R with the same probability can be estimated from the above limits. Actually, if one wished to pursue this further, a much closer (weighted) estimate could be developed such as

$$R_c = k\,R_{max}$$

where

$$k = f\left(\frac{R_{max}}{R_{min}}\right)^{*}$$

---

* Sitterly, in "LORAN" (see ref. on p.21) has proposed for the QCEP case, when converted to the notation of this mode, that

$$R \simeq \frac{0.775\,\sigma_n}{\sin\theta_x}\sqrt{1+\beta^2} \qquad [\sigma_n < \sigma_m]$$

XIII. CONSIDERATION OF UTILIZING A SINGLE SET OF DATA

As a closing observation on this model, it was felt that a converse situation concerning the use of a single data set should be discussed briefly.

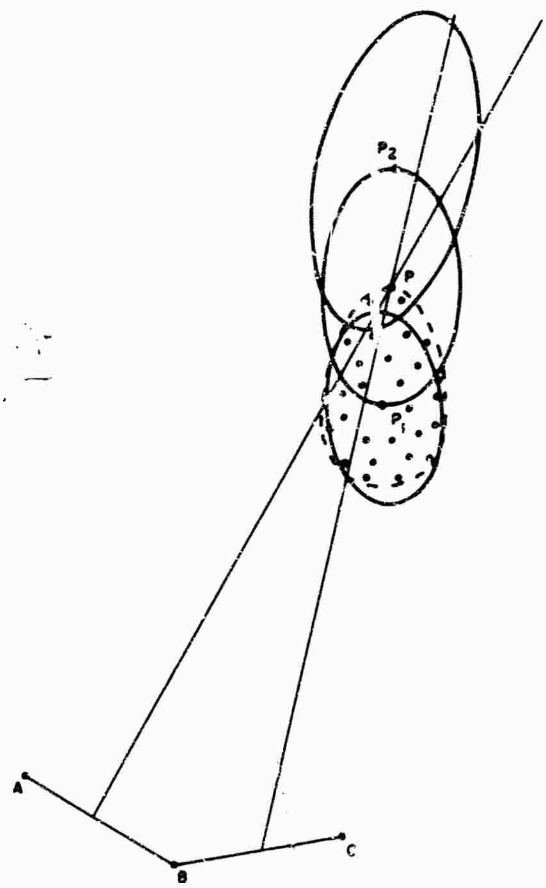


Figure 16. Effect of a Single Data Set and Ellipse Correction

Consider Figure 16. We have established a model which enables us to calculate that a signal originating at the Point P will be located by the system within or on the α = X ellipse with a probability of X. Or, succinctly, if the signal (or experiment) were repeated from P, say a million times, then 500,000 of the measured points would be within the ellipse $\alpha = 0.5$.

Suppose we are faced with a converse situation. The system receives a particular signal, and, with errors, locates it at P_1. If it is important to describe an area which will contain the true point P with a certain probability, α , what shall we do? Since we presumably have no other information available, the first impulse is to treat P_1 as a true point and construct, in the manner of this study, the ellipse for α = X as indicated in Figure 16. Another rationalization for this is that (a priori) it is just as likely that we have committed errors in one direction as in another, or there would appear to be no bias error.

Then suppose another one of these 500,000 points appears to be at P_2. This may be considered an equally probable situation because the probability density function on the periphery of the α = X ellipse is a constant. Again, we wish to describe the area which will contain P with a probability of α = X, and proceed as for P_1. Note that the ellipse about P_2 actually contains the true point while the ellipse about P_1 does not. Is this a paradox?

One possible answer would be to search for all possible true points P whose ellipses would contain P_1 as one of its (assume 500,000) measured points. A sample scattering of such true points or ellipse centers is shown. The outer limit of such a gathering would be the egg-shaped curve (shown dotted). This curve, it is suggested, comes more nearly describing the (converse) areas based on one sample measurement (P_1) which would contain the true point P with probability α. Similarly the normal α = X ellipse about P_2 would be weighted by joining centers of ellipses which contain P_2 as a peripheral point.

The mechanics for establishing this type of an inverse function and performing its solution are beyond the scope of this study. Suffice it to give one word of caution against proceeding along the lines of a singular, discrete, point solution. It must be remembered that the probability of getting precisely the point P_1, given P, is an infinitesimally small number which becomes zero when the point P_1 consumes zero dimensions. Thus a discussion or calculation of the conditional probability, p (P_1/P), appears to be meaningless. Likewise, given a measurement P_1, calculating the probability p (P/P_1) that a particular P is a true point is equally meaningless. Hence we are apparently robbed of an opportunity to analyze such expressions as,

$$p\,(P_j/P) = \frac{p\,(P_j,\ P)}{p(P)}$$

$$p\,(P/P_j) = \frac{p\,(P,\ P_j)}{p(P_j)}$$

or to apply Bayes or any other theorem. Even if we did not have this handicap, there is no a posteriori knowledge of the distribution of p(P) or $p(P_j)$ unless some sort of specific (say, target) information were available.

Fortunately the modifying or egg-shaping factor (if this be the answer to such a requirement) is a relatively small percentage of the major axis of the ellipse.

XIV. DIVERGENCE FACTOR OF THE TRUE OBLIQUE COORDINATE SYSTEM

Another and perhaps more plausible explanation lies in the basic premise or in simplification of the model. As pointed out in Appendix A Equation (A-12),

$$m = \frac{v\, r_m\, \Delta t_m}{2c \sin \theta_n}$$

The errors encountered in m, n are not (within limits prescribed for r_m, θ_m, r_n, θ_n) the cause of this so-called paradox. Further, the elliptical relationship (for three observers, Equation (G-30), Appendix G)

$$\frac{m^2}{3/2\, \sigma_m^2} + \frac{n^2}{3/2\, \sigma_n^2} - \frac{mn}{3/2\, \sigma_n \sigma_m} = s^2$$

using such calculations of m, n, is perfectly valid. However, this approximation is apparently overshadowing another assumption about which very little has been said. The grid work for the coordinates of Appendix I, Equation (I-1), (I-2), Figure I-1, etc., assumes uniform parallel lines which are directed by the LOP's or direction of the hyperbolas at the assumed true point.

As Figure 17 shows, the coordinate grid systems derived from actual conditions of Δt_m and Δt_n do not contain this ideal uniformity. Introduction of a true or non-constant grid about point P, i.e., one which follows the LOP's or hyperbolas about P, causes a variation of θ_x. Looking at Figure 17, the significant change in the mechanics of finding R is that the true R is PP'' whereas the model in using θ_x, m, and n derives PP'. In terms of formulas then, PP'' is

$$R^2 = \frac{1}{\sin^2 \theta'_x} \left[(m')^2 + (n')^2 - 2m'n' \cos \theta'_x \right] \tag{99}$$

Determination of the elliptical transforms therefore requires finding the maximum and minimum of

$$\frac{dR}{dm'} = f\,(m',\, n',\, \theta'_x) \tag{100}$$

where

$$\theta'_x = \phi(m',\, n')$$

As an (engineering) alternative to this complex problem, it is suggested that the following procedure will give a very good approximation of the true major axis values. It can be shown (Appendix K) for a given true point P and displaced LOP's that the

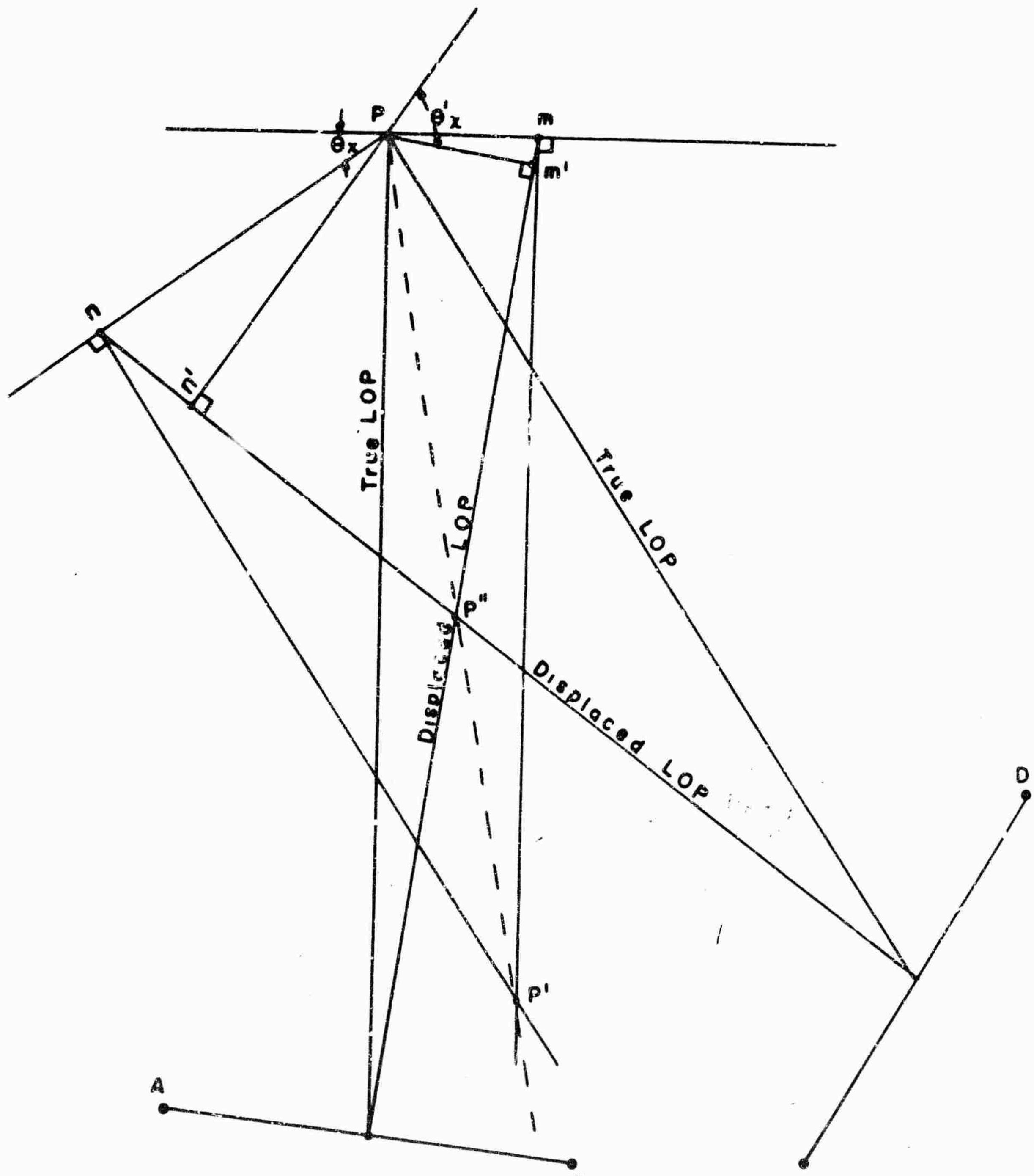


Figure 17. Comparison of Assumed and True Oblique Coordinate Systems

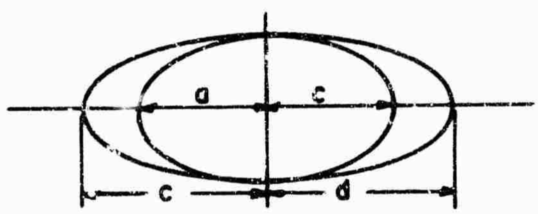
points P', P'', P, etc., all lie on a straight line. The correction factors of the original major axis are given by (see Figure K-1),

$$c/a = \frac{\theta_y}{\theta_x}$$

$$d/c = \frac{\theta_x}{\theta_z}$$

$$d/a = \frac{\theta_y}{\theta_z}$$

where c is the major axis value obtained from application of the transforms, and a and d are as shown below.



Finally, as per Appendix K, this becomes

$$c/a = \frac{\theta_x + \frac{\Delta\theta_n + \Delta\theta_m}{2}}{\theta_x} = \frac{\theta_x + \frac{n}{r_n} + \frac{m}{r_m}}{\theta_x}$$

$$d/c = \frac{\theta_x}{\theta_x - \frac{n}{r_n} - \frac{m}{r_m}}$$

$$d/a = \frac{\theta_x + \frac{n}{r_n} + \frac{m}{r_m}}{\theta_x - \frac{n}{r_n} - \frac{m}{r_m}}$$

where n and m are the required coordinate values for R_{max}. The change in the minor axis value will not be significant.

This modification or correction factor is, of course, applicable to the basic model irrespective of the so-called converse problem, i.e., it is applicable to the area about the true point P.

XV. DISCUSSION OF SYSTEMS APPLICATION WITH A DEEP SPACE FIXING EXAMPLE

While another book could be written on the applications of this basic model, comments here will be limited to introductory general observations.

With regard to the number of observers (stations) to be employed, the cost of a fourth observer must be weighed against the improvement in crossing angle (θ_x). Included in the cost factor is the problem of communicating data from the four observers over greater distances, and the inherent reduction in accuracy of four versus three observers for base lines of equal or similar length. The Γ and hence σ factors will not, as a rule, demonstrate any significant differences over a given area of concern.

The phenomenology of the transmission of the signature of the event, whether longitudinal (seismic) or transverse (light, sound, heat, radio, etc.) requires only that the velocity be known and that it be either a constant or vary with the path in a manner which is known.

One example of a sound or longitudinal wave application is the SOFAR system which locates sharp disturbances, such as the boilers exploding of a ship sinking in the sea. Standard deviation values of σ were on the order of 1.0 to 4.0 seconds depending on the ranges under consideration (up to 2,500 miles) with accuracy on the order of one square mile.

The time resolution or sharpness of the signature of the event, as stated before has direct bearing on the accuracy, e.g., Δt, σ, and σ_x, of the system. In a refined model attention must be paid to these quantities varying as a function of range. The state of the art in generating and/or measuring particular amplitude/time or frequency/time characteristics has improved tremendously over the last two to three decades. In the realm of electromagnetic disturbances, systems have been developed which obtain a σ on the order of one microsecond.

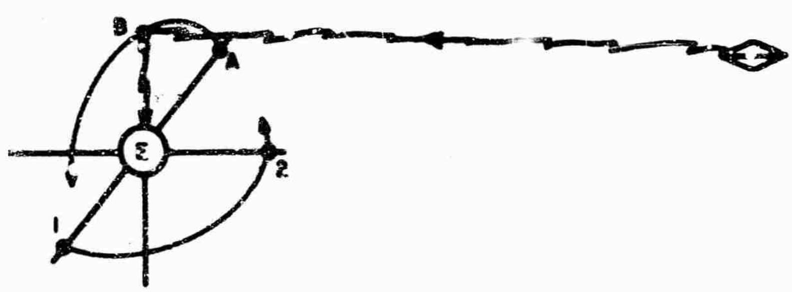
The signature must be recognized and distinguished from other (intentional or otherwise) similar signals. This challenge is certainly not peculiar to position-fixing systems.

The observers must use timing devices which are precisely synchronized with each other or which have known time displacements with respect to each other. Deviations from such conditions are charged directly to σ_x and Δt. With the development of atomic clocks with drifts of 1 part in 10^{13}, it is possible even without synchronizing communications, to stay within σ_x of 1.0 μsec.

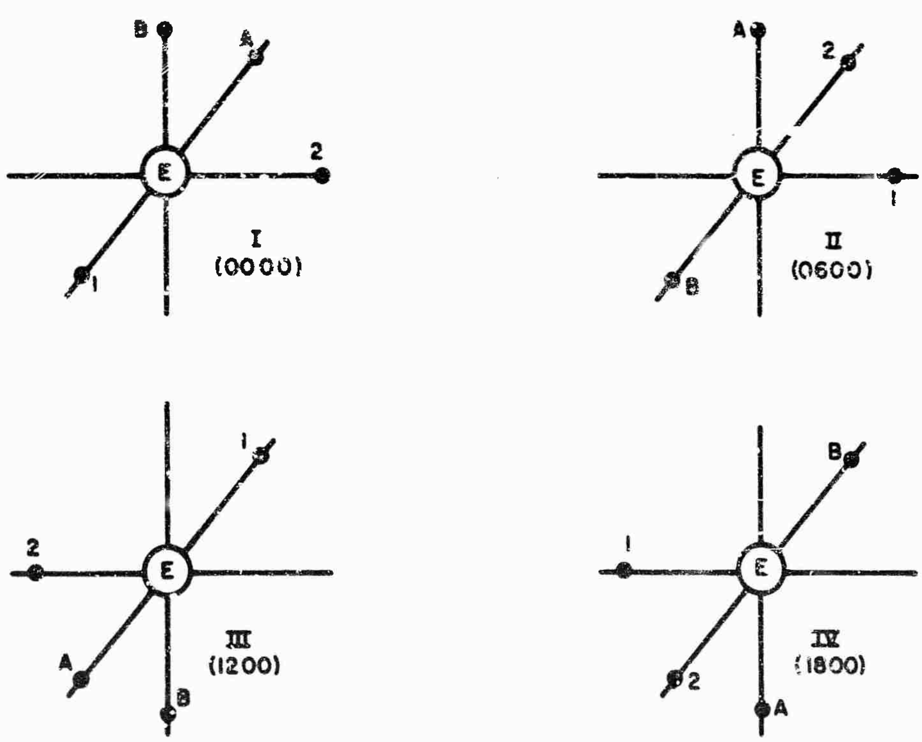
Communications must be available to transmit timing data to a common point of intelligence. This must be done reliably and fast enough to keep up with the average rate of occurrence of events.

As an ultimate in application, this model places us in a position to make direct comparisons with position-fixing systems, such as tracking space vehicles or missiles. These apparently utilize a single station radar D/F-ing principle in conjunction with multistation correlation techniques. It is the belief of the author that TOA systems will ultimately replace some D/F systems with an order of magnitude improvement in accuracy. The Air Force today has systems which show TOA to be definitely superior to D/F.

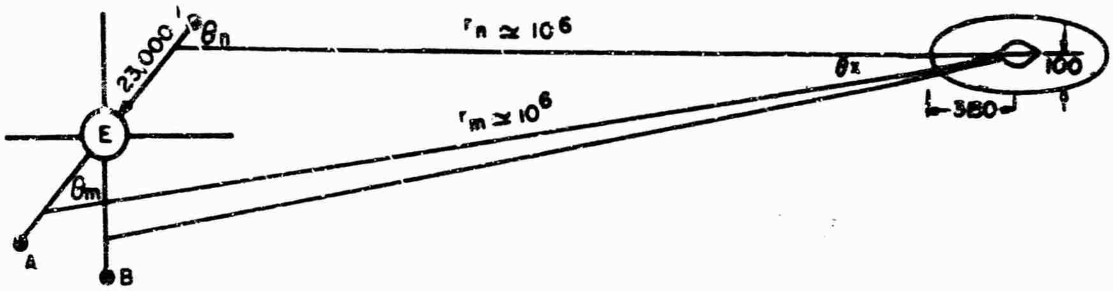
As an example, suppose we wished to reach out into deep space and measure from earth the position of a space ship such as Mariner where accuracy is extremely important. Suppose further that we have implemented the following synchronous system of four satellites where 1 and 2 are in the equatorial plane and A and B are in a polar plane.



The four coaxial states of existence which occur every six hours are



If each station is acting as an analog relay with transit time to earth known to within 0.1 microsecond, the four stations shown represent an orthogonally oriented three-dimensional measuring system. Geometrical factors for one of these states might appear as,



Assume, roughly, a range of 1 million miles and a very conservative estimate of 10 microseconds for σ_x. If we settle for an uncertainty factor of 50%, we have the following parameters

$$2C = 23,000 \text{ miles}$$

$$\sigma_x = 10 \times 10^{-6}$$

$$\alpha = 50\%$$

$$r_n \simeq r_m = 10^6$$

Obviously, θ_m and θ_n would depend upon the relative orientation of the system to the ship. However, angles of 60° and 118° are fair representations. To estimate the error magnitude in the plane of 1-A-ship

$$\Gamma_n \simeq \Gamma_m = \frac{0.188 \times 10^6 \times 10^6}{23 \times 10^3 \times 0.866} \simeq 10^7$$

and

$$\sigma_n \simeq \sigma_m = 10 \times 10^{-6} \times 10^7 = 100 \text{ miles}$$

$$\beta = 1$$

Since θ_x is a critical parameter, a close estimate is obtained by

$$\theta_x = \frac{(23,000 + 8000) \sin 60^\circ}{10^6} \text{ radians}$$

$$\theta_x = 1^\circ 30'$$

Looking up R for the QCEP on Figure 6(f) gives

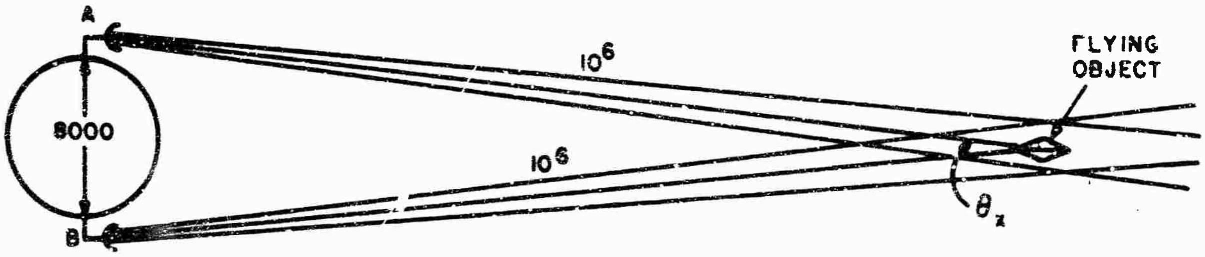
$$R = 25.5\,\sigma_3 = 25.5 \times 100 = 2,550 \text{ miles}$$

The elliptical constants in this plane were determined in conjunction with the transforms as

$$R \text{ major} = 3,150 \text{ miles}$$

$$R \text{ minor} = 100 \text{ miles}$$

To compare with the following crude yet fundamental D/F system assume antenna



deviations from the true line of propagation are of Gaussian probability density and that a standard deviation of 0.1° is representative, or

$$\sigma_\phi = 0.1^\circ$$

From this point on we can readily equate the two systems to the same model by

$$\sigma_n = \sigma_\phi r_n = 0.00175 \times 10^6 = 1.75 \times 10^3 \text{ miles}$$

$$\sigma_m = \sigma_\phi r_m = 0.00175 \times 10^6 = 1.75 \times 10^3 \text{ miles}$$

with a crossing angle of

$$\theta_x = \frac{8,000}{10^6} = 28 \text{ minutes}$$

Since the function of antenna A should be completely independent of antenna B, this system is similar to a four-observer TOA system; as a best estimate from the curves in Figure 5(f) for the plane of A-B-Ship, the QCEP is given by

$$R = 300\,\sigma_s = 1.74 \times 10^3 \times 300 = 5.25 \times 10^5 \text{ miles}$$

This figure seems unbelievable at first glance, like looking in front of our noses for the traveler. However, when we examine the facts that θ_x is less than 0.5 degree and that the beam uncertainty is $\sigma_\phi = 0.1^o$, a total collinear error of 1.0 million miles (2R) has some meaning.

If a correction of the collinear error value of R between the earth and the ship is effected in accordance with determination of θ_y (see Appendix K),we obtain a value of

$$\theta_y = \theta_x + \frac{\Delta\theta_m}{2} + \frac{\Delta\theta_n}{2} = 0.5^o + 0.5^o = 1.5^o$$

$$R' = \frac{1750.}{2\tan\theta/2} = \frac{1750.}{0.0262} = 66{,}000 \text{ miles}$$

Finally, if we wish to correct the major axis of the TOA ellipse, as per Appendix K

$$a/c = \frac{\theta_x}{\theta_x + \frac{\Delta\theta_n}{2} + \frac{\Delta\theta_m}{2}} = \frac{0.0262}{0.0262 + 2 \times \frac{61}{10^6}} = 0.995$$

$$d/c = \frac{\theta_x - \frac{\Delta\theta_n}{2} - \frac{\Delta\theta_m}{2}}{\theta_x} = \frac{0.0262 - 0.00012}{0.0262}$$

$$d/c = 1.005$$

Thus we have observed an improvement in accuracy of approximately two to three orders of magnitude. How much better this approach would be than today's refined space tracking D/F systems which may superimpose radar range or integrated CW type of data, the author has not had sufficient information to determine. There is also the additional factor that atmospheric refraction errors affect the D/F antennas but do not affect the TOA system.

APPENDIX A

MATHEMATICS OF BASIC GEOMETRIC ELEMENTS

This appendix is concerned with providing derivations of some of the more minor relationships used throughout the study.

1. Discussion of

$$m = \Gamma_m \Delta t_m \tag{A-1}$$

In Figure A-1, C is the radius of the circle and r_m the distance OP. m is the deviation of the m axis or m coordinate of the displacement of the true point P due to an error $\Delta\theta$. a is the base line intercept of the hyperbola through P. An error in the time difference of observations at A and B results in the base line error of ϵ.

m is precisely given by

$$m = r_m \tan \Delta\theta \tag{A-2}$$

For $\Delta\theta < 1^\circ$ we can write with negligible error

$$\tan \Delta\theta = \frac{ef}{C} \tag{A-3}$$

Then

$$m = r_m \frac{ef}{C} \tag{A-4}$$

In order to get ef in terms of ϵ another engineering approximation is made, with very small error, that

$$r = \theta_m \tag{A-5}$$

Then

$$\frac{ef}{\epsilon} = \frac{C}{b} \tag{A-6}$$

and

$$ef = \frac{\epsilon C}{b} \tag{A-7}$$

hence

$$m = r_m \frac{\epsilon}{b} = \frac{r_m \epsilon}{C \sin \theta_m} \tag{A-8}$$

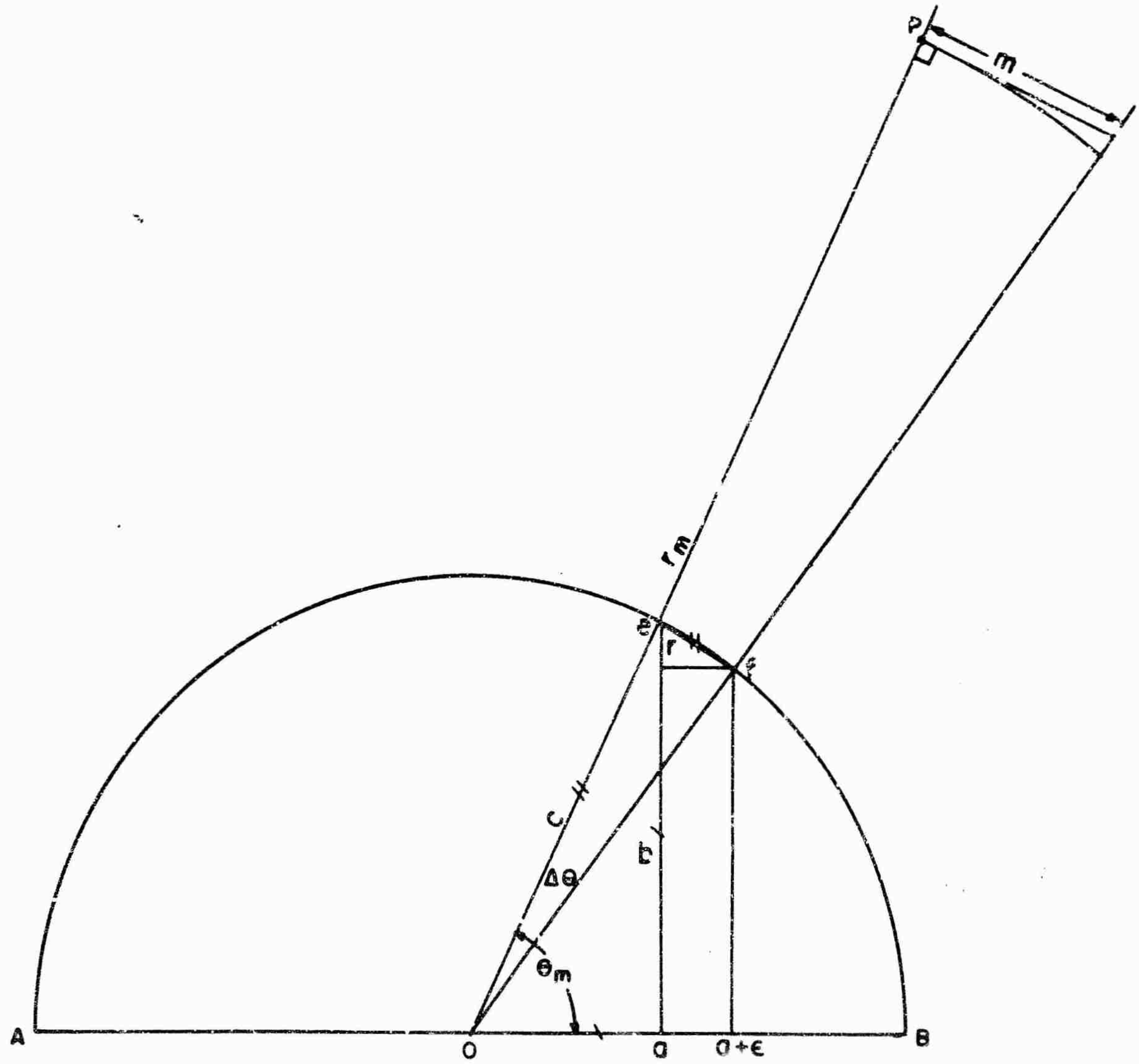


Figure A-1. Geometry of Determining m or n

Returning to the hyperbolic algebra, if ΔT_m is the net timing error made by A and B after subtraction, then

$$2(a+\epsilon) = v(T_A - T_B + \Delta t_m) \qquad \text{(A-9)}$$

Also, for perfect time readings

$$2a = v(T_A - T_B) \tag{A-10}$$

Subtracting Equation (A-10) from Equation (A-9) gives

$$\epsilon = \frac{v}{2} \Delta t_m \tag{A-11}$$

Again substituting

$$m = \frac{r_m v \Delta t_m}{2C \sin \theta_m} \tag{A-12}$$

If we choose to call

$$\Gamma_m = \frac{r_m v}{2C \sin \theta_m} \tag{A-13}$$

the final result is

$$m = \Gamma_m \Delta t_m \tag{A-14}$$

The accuracy of Equation (A-5) and hence Equation (A-12) dissolves rapidly for $\theta_{m/n} < 10^o$ or $\theta_{m/n} > 170^o$.

2. To obtain

$$\theta_m = \cos^{-1} \frac{v(t_A - t_B)}{2C} \tag{A-15}$$

As discussed in the text of this study, it may be necessary to determine θ_m and θ_n as closely as possible to either approximate a fix or analyze the error characteristics of a potential fix point. In the latter case, θ_m and θ_n would most likely be geographically measured. In the case of the former, we would be working with a given set of time measurements. In Figure A-1

$$\cos \theta_m = a/c \tag{A-16}$$

From the hyperbolic algebra

$$2a = v(t_A - t_B) \tag{A-17}$$

thus

$$\theta_m = \cos^{-1} \frac{v(t_A - t_B)}{2C}$$

Since t_A and t_B are subject to the errors discussed elsewhere, the true picture is given by

$$\theta_m = \cos^{-1} \frac{v(t_A - t_B)}{2C} + \Delta\theta_m \qquad \text{(A-18)}$$

3. To show

$$R^2 = \frac{1}{\sin^2 \theta_x} (m^2 + n^2 - 2mn \cos \theta_x) \qquad \text{(A-19)}$$

This derivation does not necessarily apply to the general problem of evaluating a point P in an oblique axis system. It may be considered a special case of an affine transformation in which the LOP displacements from the true LOP's y and z are defined to be parallel to y and z.

From Figure A-2 the general equation for the value of m is

$$m = m_o + z \sin \theta_x \qquad \text{(A-20)}$$

Further it can be seen that

$$m_o = n \cos \theta_x \qquad \text{(A-21)}$$

and

$$m = n \cos \theta_x + z \sin \theta_x$$

Solving for z

$$z = \frac{m - n \cos \theta_x}{\sin \theta_x} \qquad \text{(A-22)}$$

Also from Figure A-2

$$R^2 = n^2 + z^2 \qquad \text{(A-23)}$$

but from Equation (A-22)

$$z^2 = \frac{1}{\sin^2 \theta} (m^2 - 2mn \cos \theta_x + n^2 \cos^2 \theta_x)$$

and, substituting in Equation (A-23)

$$R^2 = \frac{1}{\sin^2 \theta_x} (n^2 \sin^2 \theta_x + m^2 + n^2 \cos^2 \theta_n - 2mn \cos \theta_x) \qquad \text{(A-24)}$$

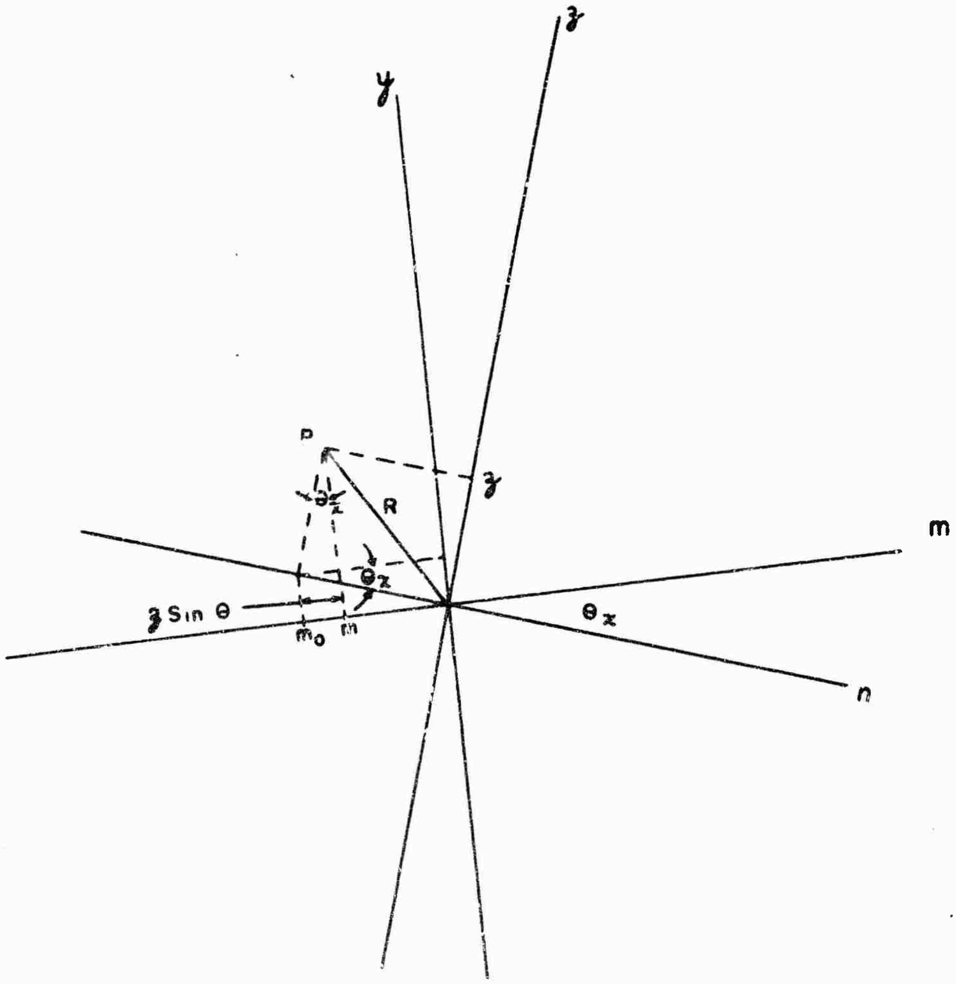


Figure A-2. Geometry of Determining R

or, finally

$$R^2 = \frac{1}{\sin^2 \theta_x} (m^2 + n^2 - 2mn \cos \theta_x) \qquad \text{(A-25)}$$

APPENDIX B

QCEP CALCULATIONS FOR UNIFORM ERROR PROBABILITY DISTRIBUTION

This is an analysis of areas since the probability of a fix occurring in a given area is directly proportional to that area. Figures B-1, B-2, and B-3 illustrate the three possible cases. Quantitatively stated these are:

CASE I $\quad 2R < \Delta S_n < \Delta S_m$

CASE II $\quad \Delta S_n < 2R < \Delta S_m$

CASE III $\quad \Delta S_n < \Delta S_m < 2R$

The difficulty in this analysis lies in formulating that portion of the (QCEP) circle which lies within the error quadrangle.

Case I is the simplest since all of the circle is contained within the quadrangle. Assuming $\Delta S_n < \Delta S_m$ for the area of the quadrangle

$$A_1 = \frac{\Delta S_n \Delta S_m}{\sin \theta_x} \tag{B-1}$$

For the circle

$$A_2 = \pi R^2$$

Then for a QCEP of 50%

$$A_2 = 1/2 \, A_1 \tag{B-2}$$

$$\pi R^2 = \frac{1}{2} \frac{\Delta S_n \Delta S_m}{\sin \theta_x} \tag{B-3}$$

Solving for R

$$R = \sqrt{\frac{\Delta S_n \Delta S_m}{2\pi \sin \theta_x}} \tag{B-4}$$

Since $\quad \Delta S_n \leq \Delta S_m$

assume $\quad \Delta S_m = \gamma \Delta S_n$

where $\quad \gamma = 1$

$$R = \sqrt{\frac{\gamma \Delta S_n^2}{2\pi \sin \theta_x}} = \frac{\Delta S_n}{2} \sqrt{\frac{2\gamma}{\pi \sin \theta_x}} \tag{B-5}$$

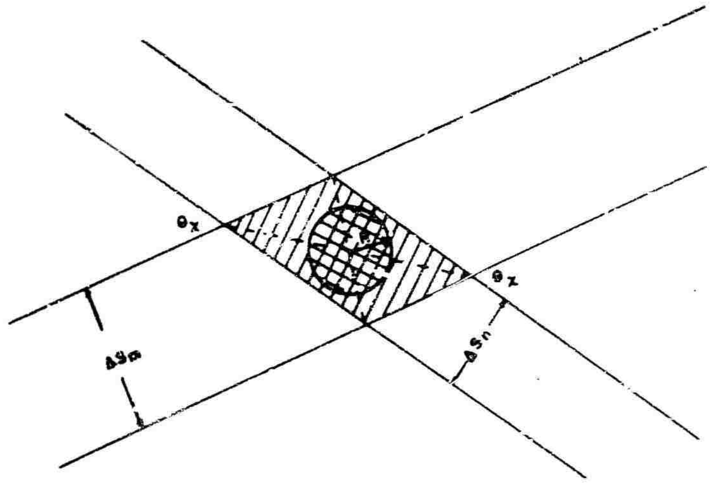


Figure B-1. QCEP Circle - Case 1

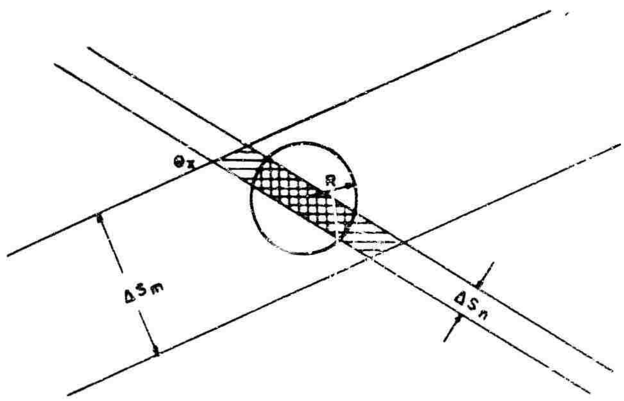


Figure B-2. QCEP Circle - Case II

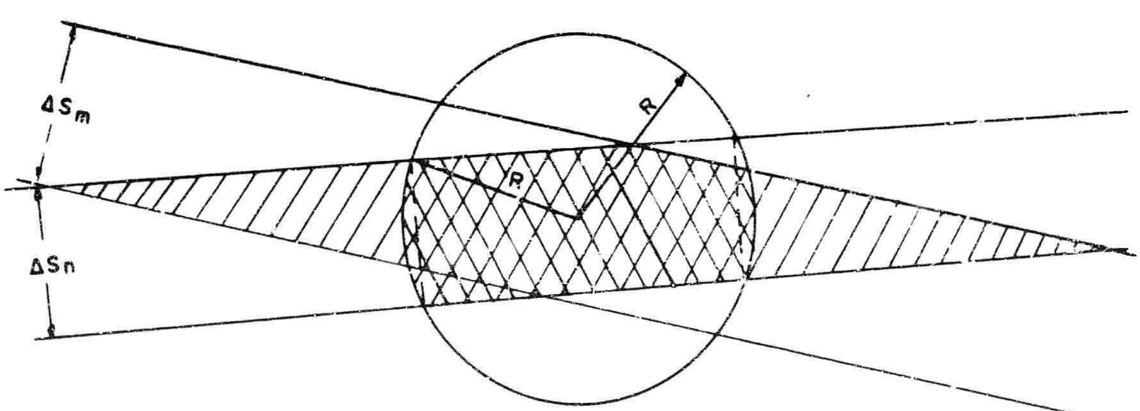


Figure B-3. QCEP Circle - Case III

But since we have the requirement

$$R \leq \frac{\Delta S_n}{2}$$

the result is the requirement

$$\sqrt{\frac{2\gamma}{\pi \sin \theta_x}} \leq 1 \qquad \text{(B-6)}$$

or

$$\frac{\gamma}{\sin \theta} \leq \pi/2$$

This results in the limits

$$\begin{cases} \gamma = 1 \\ \theta > 39° 35' \end{cases} \quad \text{and} \quad \begin{cases} \gamma < 1.57 \\ \theta = 90° \end{cases} \qquad \text{(B-7)}$$

Returning to Equation (B-4) and from Appendix A

$$\Delta S_i = r_i \Delta \theta_i = \frac{v_i \epsilon}{C \sin \theta_i} \qquad \text{(B-8)}$$

By substitution

$$R = \sqrt{\frac{1}{2\pi \sin \theta_x} \times \frac{r_n \epsilon}{C \sin \theta_n} \times \frac{r_m \epsilon}{C \sin \theta_m}}$$

$$R = \frac{\epsilon}{C} \sqrt{\frac{r_n r_m}{2\pi \sin \theta_x \sin \theta_n \sin \theta_m}} \qquad \text{(B-9)}$$

For Case II

$$\Delta S_n < 2R < \Delta S_m$$

$$A_2 \simeq 2R \, \Delta S_n = \frac{A_1}{2} \qquad \text{(B-10)}$$

$$2R \, \Delta S_n = \frac{\Delta S_n \, \Delta S_m}{2 \sin \theta_x} \qquad \text{(B-11)}$$

$$R = \frac{\Delta S_m}{4 \sin \theta_x} \qquad \text{(B-12)}$$

Finally

$$R = \frac{r_n \epsilon}{4C \sin \theta_x \sin \theta_n} \qquad \text{(B-13)}$$

CASE III $\Delta S_n < \Delta S_m < 2R$

This is of interest because it embraces crossing angles of less than 39° which is generally an area of fix points with range on the order of 1.8 base lines and beyond.

As observed in Figure B-3, a close approximation to the enclosed area is given by the area contained in the rectangle

$$\Delta S_n \text{ by } \sqrt{4R^2 - \Delta S_n^{\,2}} \qquad \text{(B-14)}$$

or

$$A_2 = \Delta S_n \sqrt{4R^2 - \Delta S_n^{\,2}}$$

and since $A_2 = \frac{A_1}{2}$

$$\Delta S_n \sqrt{4R^2 - \Delta S_n^{\,2}} = \frac{\Delta S_n \, \Delta S_m}{2 \sin \theta_x} \qquad \text{(B-15)}$$

Solving for R

$$R = \frac{1}{2} \sqrt{\frac{\Delta S_m^{\,2}}{4 \sin^2 \theta_x} + \Delta S_n^{\,2}} \qquad \text{(B-16)}$$

By substitution via Equation (B-8)

$$R = \frac{\epsilon}{2C} \sqrt{\frac{r_m^{\,2}}{4 \sin^2 \theta_x \sin^2 \theta_m} + \frac{r_n^{\,2}}{\sin^2 \theta_n}} \qquad \text{(B-17)}$$

APPENDIX C

QCEP DERIVATIONS FOR A NORMAL PROBABILITY DISTRIBUTION OF ERRORS FOR FOUR OBSERVERS

As stated in the text, the probability of a point P' lying within a circle of radius R is given by

$$\alpha = \frac{1}{2\pi \sigma_n \sigma_m} \int \int_A \exp\left[\frac{-n^2}{2\sigma_n^2}\right] \exp\left[-\frac{m^2}{2\sigma_m^2}\right] dmdn \qquad \text{(C-1)}$$

where A is the circle of radius R. σ_n and σ_m are system standard deviations as defined elsewhere.

The immediate problem in performing the indicated double integration is that we are not dealing with orthogonal axes in which one variable can be held constant at some known value while summation is performed throughout the range of the other variable. In order to utilize this accepted method of performing double integration one of the variables must be reduced to a component which is orthogonal to the other variable.

In Figure C-1 assume the circle R provides the desired probability α and let z be an axis at right angles to n. Any point p on a line parallel to z will have a constant value of n, such as

$$n = n_K$$

It was shown in Appendix A that the relationship between the quantities n, m, and θ_x is given by

$$m = n \cos \theta_x + z \sin \theta_x$$

If this substitution is made for m in Equation (A-1) and n is chosen as some constant value, e.g., n_K, then variations in m become a function of variations in z alone, which in turn satisfies the orthogonal requirement for integration, i.e.

$$m = n_K \cos \theta_x + z \sin \theta_x \qquad \text{(C-2)}$$

and

$$dm = dz \sin \theta_x \qquad \text{(C-3)}$$

Substituting in Equation (C-1) gives

$$\alpha = \frac{1}{2\pi \sigma_n \sigma_m} \int \int_A \exp\left[\frac{-n^2}{2\sigma_n^2}\right] \exp\left[\frac{-(n_K \cos \theta_x + z \sin \theta_x)^2}{2\sigma_m^2}\right] dndz \sin \theta_x \qquad \text{(C-4)}$$

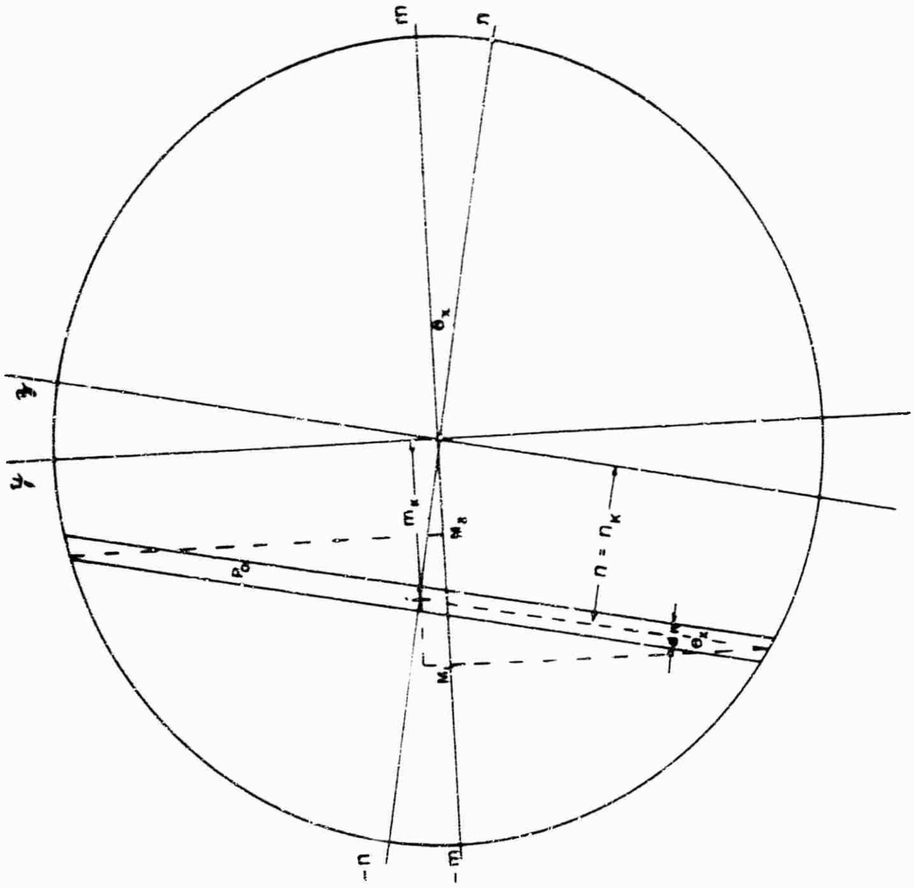


Figure C-1. Elements of QCEP Geometry

It can be seen further from Figure C-1 that the limits of integration are

$$-R \leq n_K \leq +R$$

$$-\sqrt{R^2-n_K^{\ 2}} \leq z \leq +\sqrt{R^2-n_K^{\ 2}}$$

Since we are now performing the integration first with respect to z and then with respect to n, the subscript K can be dropped.

$$\alpha = \frac{\sin\theta}{2\pi\,\sigma_n\,\sigma_m} \int_{-R}^{R} \int_{-\sqrt{R^2-n^2}}^{+\sqrt{R^2-n^2}} \exp\left[\frac{-n^2}{2\sigma_n^{\ 2}}\right] \exp\left[\frac{-(n\cos\theta + z\sin\theta)^2}{2\,\sigma_m^{\ 2}}\right] dz\, dn \qquad \text{(C-5)}$$

For verification purposes it is convenient to rewrite the last term as

$$\exp\left[\frac{-\sin^2\theta(n\cot\theta+z)^2}{2\sigma_m^2}\right]$$

and

$$\alpha = \frac{\sin\theta}{2\pi\,\sigma_m\,\sigma_n}\int\int_{-R-\sqrt{R^2-n^2}}^{R+\sqrt{R^2-n^2}}\exp\left[\frac{-n^2}{2\sigma_n^2}\right]\exp\left[\frac{-\sin^2\theta\,(n\cot\theta+z)^2}{2\sigma_m^2}\right]dz\,dn \qquad \text{(C-6)}$$

To prove the correctness of this analysis, it is necessary that for an infinite circle all possibilities are covered and

$$\alpha = 1$$

But Equation (C-6) becomes

$$\alpha = \frac{\sin\theta}{2\pi\,\sigma_n\,\sigma_m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-n^2}{2\sigma_n^2}\right]\exp\left[\frac{-\sin^2\theta(n\cot\theta+z)^2}{2\sigma_m^2}\right]dz\,dn = 1 \qquad \text{(C-7)}$$

Since these two integrations are to be performed independently, in any order, and since it is easily shown that

$$\frac{1}{2\pi\,\sigma_n}\int_{-\infty}^{\infty}\exp\left[\frac{-n^2}{2\sigma^2}\right]dn = 1 \qquad \text{(C-8)}$$

it is also necessary that

$$\frac{\sin\theta}{\sqrt{2\pi}\,\sigma_m}\int_{-\infty}^{\infty}\exp\left[\frac{-\sin^2\theta\,(n\cot\theta+z)^2}{2\sigma_m^2}\right]dz = 1 \qquad \text{(C-9)}$$

To check, let

$$v^2 = (n\cot\theta+z)^2$$

$$dv = dz$$

when

$$z = \infty\,,\quad v = \infty$$

and Equation (C-9) becomes

$$\frac{\sin\theta}{\sqrt{2\pi}\,\sigma_m}\int_{-\infty}^{\infty}\exp\left[\frac{\sin^2\theta}{2\sigma_m}v\right]dv = \frac{\sin\theta}{\sqrt{2\pi}\,\sigma_m}\sqrt{\frac{\pi}{\frac{\sin^2\theta}{2\sigma_m^2}}} = 1$$

Thus the necessary condition is met.

Another way of stating this proposition, which is more general than Equation (C-6), is to consider that every N_k, m (as shown in Figure C-1) traverses the values from M_1 to M_2, and to step from the idea of an orthogonal component to one of summing all the probabilities associated with combinations of m_i by n_k as m_i goes from M_1 to M_2, i.e.

$$\alpha = \frac{1}{2\pi\, \sigma_n \sigma_m} \int_{-R}^{R} \int_{M_1}^{M_2} \exp\left[\frac{-n^2}{2\sigma_n^2}\right] \exp\left[\frac{-m^2}{2\sigma_n^2}\right] dndm \qquad \text{(C-10)}$$

where

$$M_1 = f_1(n) = n_k \cos\theta_x - z \sin\theta_x$$

$$M_2 = f_2(n) = n_k \cos\theta_x + z \sin\theta_x$$

or (dropping the subscript)

$$f_1(n) = n \cos\theta_x - \sqrt{R^2 - n^2}\, \sin\theta_x$$

$$f_2(n) = n \cos\theta_x + \sqrt{R^2 - n^2}\, \sin\theta_x$$

and

$$\alpha = \frac{1}{2\pi\, \sigma_n \sigma_m} \int_{-R}^{R} \int_{f_1(n)}^{f_2(n)} \exp\left[\frac{-n^2}{2\sigma_n^2}\right] \exp\left[\frac{-m^2}{2\sigma_m^2}\right] dmdn \qquad \text{(C-11)}$$

which is the formula quoted in the text.

Since this concept deals with the well-known Error Function or Fehlerintegral for which there is no antiderivative, and since from a system aspect we may be dealing with small values of θ_x, it is desirable from an engineering standpoint to try to reduce this to a single integral even if only for limited conditions. One way of performing such an approximation is to consider the integration of f(m) first, e.g.

$$\xi = \frac{1}{\sqrt{2\pi}\, \sigma_n} \int_{f_1(n)}^{f_2(n)} \exp\left[\frac{-m^2}{2\sigma_m^2}\right] dm \qquad \text{(C-12)}$$

Referring to the normal probability curve of Figure C-2, the values of $f_1(n)$ and $f_2(n)$ are represented as M_1 and M_2. Since performing Equation (C-12) is equivalent to finding the prescribed area under this curve, ξ is equivalent to the shaded area between M_1 and M_2. This area can be approximated in various ways such as performing a series expansion about the point $(M_2 + M_1)/2$. This, however, tends to complicate the situation. Therefore, it was decided that for small values of R sin θ_x

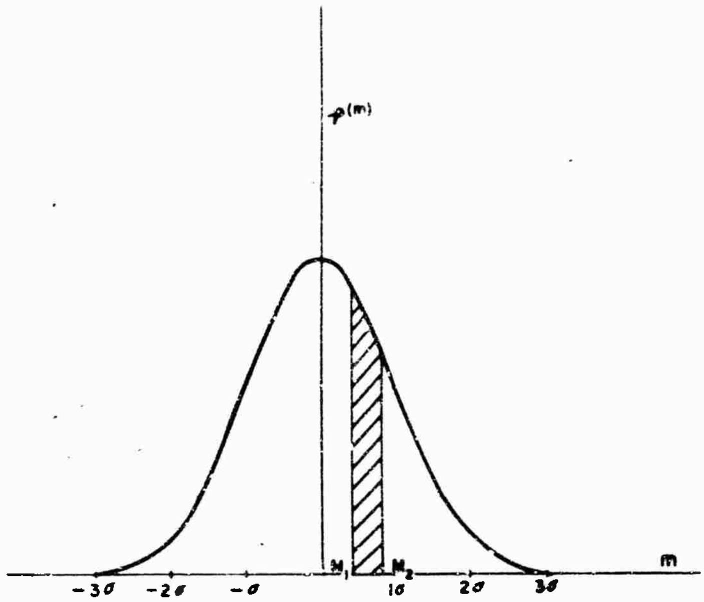


Figure C-2. A Normal Distribution Interpretation

a reasonable approximation is obtained by assuming the straight line between $p(M_1)$ and $p(M_2)$. The area is then the rectangle

$$(M_2 - M_1)\exp\left[\frac{-M_2^2}{2\,\sigma_m^2}\right]$$

plus the area of the triangle

$$\frac{1}{2}(M_2 - M_1)\left(\exp\left[\frac{-M_1^2}{2\sigma_n^2}\right] - \exp\left[\frac{-M_2^2}{2\sigma_m^2}\right]\right)$$

and Equation (C-12) becomes

$$\xi = \frac{1}{2\pi\,\sigma_m}\left\{(M_2-M_1)\exp\left[\frac{-M_2^2}{2\sigma_n^2}\right] + \frac{1}{2}(M_2-M_1)\left(\exp\left[\frac{-M_1^2}{2\sigma_m^2}\right] - \exp\left[\frac{-M_2^2}{2\sigma_m^2}\right]\right)\right\}$$

Expanding and collecting terms

$$\xi = \frac{1}{4\pi\,\sigma_m}(M_2-M_1)\left(\exp\left[\frac{-M_1^2}{2\,\sigma_m^2}\right] + \exp\left[\frac{-M_2^2}{2\,\sigma_m^2}\right]\right)$$

Substituting into Equation (C-11) and replacing M

$$\alpha = \frac{1}{2\pi\,\sigma_n\,\sigma_m} \int_{-R}^{R} \left\{f_2(n) - f_1(n)\right\} \left\{ \exp\left[\frac{-f_1^{\,2}(n)}{2\,\sigma_m^{\,2}}\right] + \exp\left[\frac{-f_2^{\,2}(n)}{2\,\sigma_m^{\,2}}\right] \right\} dn \tag{C-13}$$

which is the formula quoted in the text.

The limiting value of R sin θ_x is somewhat arbitrary. It can be seen from Figures C-1 and C-2 that the worst error is committed in the region of m = 0 for here

$$M_2 - M_1 \simeq 2R \sin \theta_x$$

If it is desired to hold this value to, e.g.,

$$M_2 - M_1 = b\sigma_m$$

then

$$R \sin \theta_x \leq \frac{b\,\sigma_m}{2} \tag{C-14}$$

While this engineering limit has not been fully explored in this study, it appears from the computer runs that good confirmation with the general formula is obtained up to a Δm of at least one sigma or b = 1.

Returning to Equation (C-11) and a more general evaluation of α which is rigorous for all θ_x, we proceed by using the formula (C-6) as follows. It is first desired to evaluate

$$\lambda = \int_{-\sqrt{R^2-n^2}}^{+\sqrt{R^2-n^2}} \exp\left[\frac{-\sin^2\theta\,(n \cot\theta + z)^2}{2\,\sigma_m^{\,2}}\right] dn$$

Let

$$u^2 = (n \cot \theta + z)^2$$

$$du = dz$$

For limits

when z =	u =
$-\sqrt{R^2-n^2}$	$n \cot \theta - \sqrt{R^2-n^2}$
$+\sqrt{R^2-n^2}$	$n \cot \theta + \sqrt{R^2-n^2}$

and λ becomes

$$\lambda = \int_{n\cot\theta - \sqrt{R^2-n^2}}^{n\cot\theta + \sqrt{R^2-n^2}} \exp\left[-\frac{\sin^2\theta\, u^2}{2\sigma_m^2}\right] du$$

which can also be written

$$\lambda = \int_{o}^{n\cot\theta + \sqrt{R^2-n^2}} \exp\left[-\frac{\sin^2\theta\, u^2}{2\sigma_m^2}\right] du$$

$$- \int_{o}^{n\cot\theta - \sqrt{R^2-n^2}} \exp\left[-\frac{\sin^2\theta\, u^2}{2\sigma_m^2}\right] du$$

Another substitution

$$v^2 = \frac{\sin^2\theta}{2\sigma_m^2} u^2$$

$$du = \frac{dv}{\sqrt{\frac{\sin^2\theta}{2\sigma_m^2}}} = \frac{\sqrt{2}\,\sigma_m}{\sin\theta} \quad dv$$

and for limits

u	v
$n\cot\theta + \sqrt{R^2-n^2}$	$\frac{\sin\theta}{\sqrt{2}\,\sigma_m}\left(n\cot\theta + \sqrt{R^2-n^2}\right) = V_1$
$n\cot\theta - \sqrt{R^2-n^2}$	$\frac{\sin\theta}{\sqrt{2}\,\sigma_m}\left(n\cot\theta - \sqrt{R^2-n^2}\right) = V_2$

and we again rewrite λ as

$$\lambda = \frac{\sqrt{2}\,\sigma_m}{\sin\theta}\left\{\int_{o}^{V_1}\exp\left[-v^2\right] dv - \int_{o}^{V_2}\exp\left[-v^2\right] dv\right\} \qquad \text{(C-15)}$$

If the coefficient is rewritten as

$$\frac{\sqrt{2}\,\sigma_m}{\sin\theta} = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{2\pi}\,\sigma_m}{2\sin\theta}$$

$$\lambda = \frac{\sqrt{2\pi}\,\sigma_m}{2\sin\theta}\left\{\frac{2}{\pi}\int_0^{V_1}\exp\left[-v^2\right]dv - \frac{2}{\pi}\int_0^{V_2}\exp\left[-v^2\right]dv\right\} \qquad \text{(C-16)}$$

By definition the Error integral or Fehlerintegral is given by

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^{x}\exp\left[-t^2\right]dt \qquad \text{(C-17)}$$

Therefore

$$\lambda = \frac{\sqrt{2\pi}\,\sigma_m}{2\sin\theta}\left[\Phi(v_1) - \Phi(v_2)\right] \qquad \text{(C-18)}$$

Finally, returning to Equation (C-6)

$$\alpha = \frac{1}{2\sqrt{2\pi}\,\sigma_n}\int_{-R}^{R}\left[\Phi(v_1) - \Phi(v_2)\right]\exp\left[\frac{-n^2}{2\sigma_n^2}\right]dn \qquad \text{(C-19)}$$

Since this is not in a form compatible with graphical or computer solution, the following calculations are made.

Recall that

$$V_1 = \frac{\sin\theta}{\sqrt{2}\,\sigma_m}\left(n\cot\theta + \sqrt{R^2 - n^2}\right)$$

Factoring out an R gives

$$V_1 = \frac{R\sin\theta}{\sqrt{2}\,\sigma_m}\left(\frac{n}{R}\cot\theta + \sqrt{1 - (n/R)^2}\right) \qquad \text{(C-20)}$$

Let

$$\sigma_m^2 = KR^2$$

$$x = n/R$$

$$dn = R\,dx$$

then

$$V_1 = \frac{\sin\theta}{\sqrt{2K}}\left(x \cot\theta + \sqrt{1-x^2}\right)$$

Similarly

$$V_2 = \frac{\sin\theta}{\sqrt{2K}}\left(x \cot\theta - \sqrt{1-x^2}\right)$$

then

$$\Phi(V_1) = \Phi\left\{\frac{\sin\theta}{\sqrt{2K}}\left(x \cot\theta + \sqrt{1-x^2}\right)\right\} = \Phi_1(x) \tag{C-21}$$

and

$$\Phi(V_2) = \Phi\left\{\frac{\sin\theta}{\sqrt{2K}}\left(x \cot\theta - \sqrt{1-x^2}\right)\right\} = \Phi_2(x) \tag{C-22}$$

Using β as previously defined, $\beta = \sigma_n/\sigma_m$, we can also write

$$\exp\left[-\frac{n^2}{2\sigma_n^2}\right] = \exp\left[-\frac{n^2}{2\beta^2\sigma_m^2}\right] = \exp\left[-\frac{n^2}{2\beta^2 R^2 K}\right]$$

$$= \exp\left[-\frac{x^2}{2K\beta^2}\right]$$

and Equation (C-19) becomes

$$\alpha = \frac{R}{2\sqrt{2\pi}\,\sigma_n}\int_{-1}^{1}\left[\Phi_1(x) - \Phi_2(x)\right]\exp\left[-\frac{x^2}{2K\beta^2}\right]dx \tag{C-23}$$

Substitution of $\sigma_n = \beta\,\sigma_m$ and $\sigma_m/R = \sqrt{K}$ into this coefficient completes the transformation and

$$\alpha = \frac{1}{2\beta\sqrt{2\pi K}}\int_{-1}^{1}\left[\Phi_1(x) - \Phi_2(x)\right]\exp\left[-\frac{x^2}{2K\beta^2}\right]dx \tag{C-24}$$

This is the formula quoted in the text.

A word on symmetry. If, in the original solution of α, the initial integration had been performed for the variable n instead of m, the result would have been

$$\alpha = \frac{1}{2\sigma_m\sqrt{2\pi}} \int_{-1}^{+1} \exp\left[-\frac{m^2}{2\sigma_m^2/R}\right] \left\{ \Phi\left[\frac{\sin\theta}{\sqrt{2}\,\sigma_n/R}\left(\frac{m}{R}\cot\theta + \sqrt{1-(\frac{m}{R})^2}\right)\right] \right.$$

$$\left. - \Phi\left[\frac{\sin\theta}{\sqrt{2}\,\sigma_n/R}\left(\frac{m}{R}\cot\theta - \sqrt{1-(\frac{m}{R})^2}\right)\right] \right\} dm \qquad \text{(C-25)}$$

If the choice had been made to let

$$x = m/R$$

$$K = \sigma_n^2/R^2$$

$$\beta = \sigma_m/\sigma_n$$

the result would have been

$$\alpha = \frac{1}{2\beta\sqrt{2\pi K}} \int_{-1}^{1} \exp\left[-\frac{x^2}{2\beta^2 K}\right] \left\{ \Phi\left[\frac{\sin\theta}{\sqrt{2K}}\left(x\cot\theta + \sqrt{1-x^2}\right)\right] \right.$$

$$\left. - \Phi\left[\frac{\sin\theta}{\sqrt{2K}}\left(x\cot\theta - \sqrt{1-x^2}\right)\right] \right\} dx \qquad \text{(C-26)}$$

Comparing this with the results of Equations (C-21), (22), (23), and (24), it is readily seen that if the values of K and β in the first solution are equal to the parameters K and β in the second solution, identical values of α will result. Thus we have a choice, shown in the table in the text, which permits eliminating the plotting of any $\beta < 1$.

The abscissae are double scaled to give α versus both K and R where R is in terms of whichever σ is the lesser, i.e.

$$R = \sigma_s/\sqrt{K}$$

For example, if

$$\sigma_n > \sigma_m, \; \sigma_s = \sigma_m$$

It is hoped that enough data is presented in the curves of Figure 5 to permit good engineering estimates for any system requiring the QCEP method of procedure.

APPENDIX D

QCEP DERIVATIONS FOR A NORMAL PROBABILITY DISTRIBUTION OF ERROR FOR THREE OBSERVERS

It is shown in Appendix J that the joint probability density function for three observers is

$$f(n, m) = \frac{1}{\sqrt{3}\,\pi\,\sigma_n\,\sigma_m} \exp\left[-\frac{2}{3}\left(\frac{n^2}{\sigma_n^2} + \frac{m^2}{\sigma_m^2} - \frac{mn}{\sigma_n\,\sigma_m}\right)\right] \qquad \text{(D-1)}$$

The total summation of the probabilities associated with each point m, n is the value of α given by

$$\alpha = \frac{1}{\sqrt{3}\,\pi\,\sigma_n\,\sigma_m} \int_{-R}^{R} \int_{m_1}^{m_2} \exp\left[-\left(\frac{n^2}{(3/2)\sigma_n^2} + \frac{m^2}{(3/2)\sigma_m^2} - \frac{mn}{(3/2)\sigma_m\,\sigma_n}\right)\right] dmdn \qquad \text{(D-2)}$$

where

$$m_1 = f_1(n) = n \cos\theta_x - \sqrt{R^2 - n^2}\,\sin\theta_x \qquad \text{(D-3)}$$

$$m_2 = f_2(n) = n \cos\theta_x + \sqrt{R^2 - n^2}\,\sin\theta_x \qquad \text{(D-4)}$$

Let

$$u^2 = \frac{n^2}{(3/2)\sigma_n^2}$$

$$v^2 = \frac{m^2}{(3/2)\sigma_m^2}$$

Since we can start the process by holding n constant at some value, e.g., n_K

$$\alpha = \frac{1}{\sqrt{3}\,\pi\,\sigma_n\,\sigma_m} \int_{-R'}^{R'} \int_{v_1}^{v_2} \exp\left[-(u^2 + v^2 - uv)\right] \times (3/2)\sigma_n\,\sigma_m \, du\, dv \qquad \text{(D-5)}$$

where

$$v_1 = \frac{m_1}{\sqrt{3/2}\,\sigma_m}$$

$$v_2 = \frac{m_2}{\sqrt{3/2}\,\sigma_m}$$

or

$$\alpha = \frac{\sqrt{3}}{2\pi} \int_{\frac{-R}{\sqrt{3/2}\,\sigma_n}}^{\frac{R}{\sqrt{3/2}\,\sigma_n}} \int_{v_1}^{v_2} \exp\left[-(u^2 + v^2 - uv)\right] du dv \qquad \text{(D-6)}$$

Therefore, holding u constant

$$\alpha = \frac{\sqrt{3}}{2\pi} \int_{-R'}^{R'} du \left\{ \int_{v_1}^{v_2} \exp\left[-(v^2 - uv)\right] \right\} \exp\left[-u^2\right] dv \qquad \text{(D-7)}$$

To evaluate the first integral

$$\lambda = \int_{v_1}^{v_2} \exp\left[-(v^2 - uv)\right] dv \qquad \text{(D-8)}$$

proceed by completing square

$$\lambda = \int_{v_1}^{v_2} \exp\left[-(v^2 - uv + \frac{u^2}{4} - \frac{u^2}{4})\right] dv \qquad \text{(D-9)}$$

$$= \exp\left[+\frac{u^2}{4}\right] \int_{v_1}^{v_2} \exp\left[-(v - \frac{u}{2})^2\right] dv$$

Let

$$y^2 = (v - \frac{u}{2})^2$$

$$dy = dv$$

$$\lambda = \exp\left[+\frac{u^2}{4}\right] \int_{y_1}^{y_2} \exp\left[-y^2\right] dy \qquad \text{(D-10)}$$

Returning to α and rewriting

$$\alpha = \frac{\sqrt{3}}{4\sqrt{\pi}} \int_{-R'}^{R'} \left\{ \int_{y_1}^{y^2} \frac{2}{\pi} \exp\left[-y^2\right] dy \right\} \exp\left[-\frac{3}{4} u^2\right] du \qquad \text{(D-11)}$$

which by definition of $\Phi(y)$ is

$$\alpha = \frac{1}{4}\sqrt{\frac{3}{\pi}} \int_{-R'}^{R'} [\Phi(y_2) - \Phi(y_1)] \exp\left[-\frac{3}{4}u^2\right] du \tag{D-12}$$

Reversing the substitution process

$$\alpha = \frac{1}{4}\sqrt{\frac{3}{\pi}} \int_{-R'}^{R'} [\Phi(v_2 - \frac{u}{2}) - \Phi(v_1 - \frac{u}{2})] \exp\left[-3/4\, u^2\right] du \tag{D-13}$$

$$v_2 = \frac{1}{\sqrt{3/2}\,\sigma_m} m_2 = \frac{1}{\sqrt{3/2}\,\sigma_m} (n \cos\theta + \sqrt{R^2 - n^2} \sin\theta)$$

$$v_1 = \frac{1}{\sqrt{3/2}\,\sigma_m} (n \cos\theta - \sqrt{R^2 - n^2} \sin\theta$$

$$\alpha = \frac{1}{4}\sqrt{\frac{3}{\pi}} \int_{-R'}^{R'} [\Phi\left\{\frac{1}{\sqrt{3/2}\,\sigma_m}(n\cos\theta + \sqrt{R^2-n^2}\sin\theta) - \frac{u}{2}\right\}$$

$$- \Phi\left\{\frac{1}{\sqrt{3/2}\,\sigma_m}(n\cos\theta - \sqrt{R^2-n^2}\sin\theta) - \frac{u}{2}\right\} \exp[-(3/4)u^2]\, du \tag{D-14}$$

And, again for u,

$$\alpha = \frac{1}{4}\sqrt{\frac{3}{\pi}} \int_{-R}^{R} \left\{ \Phi\left[\frac{1}{\sqrt{3/2}\,\sigma_m}\left(n\cos\theta + \sqrt{R^2-n^2}\sin\theta\right) - \frac{n}{2\sqrt{3/2}\,\sigma_n}\right]\right. \tag{D-15}$$

$$\left. - \Phi\left[\frac{1}{\sqrt{3/2}\,\sigma_m}\left(n\cos\theta - \sqrt{R^2-n^2}\sin\theta\right) - \frac{n}{2\sqrt{3/2}\,\sigma_n}\right]\right\} \exp\left[-\frac{3}{4}\frac{n^2}{(3/2)\sigma_n^2}\right] \frac{dn}{\sqrt{3/2}\,\sigma_n}$$

$$\alpha = \frac{1}{2\sqrt{2\pi}\,\sigma_n} \int_{-R}^{R} \Phi\left[\frac{1}{\sqrt{3/2}\,\sigma_m}\left(n\cos\theta + \sqrt{R^2-n^2}\sin\theta\right) - \frac{n}{2\sqrt{3/2}\,\sigma_n}\right] \tag{D-16}$$

$$- \Phi\left[\frac{1}{\sqrt{3/2}\,\sigma_m}\left(n\cos\theta - \sqrt{R^2-n^2}\sin\theta\right) - \frac{n}{2\sqrt{3/2}\,\sigma_n}\right]\Bigg\} \exp\left[-\frac{n^2}{2\sigma_n^2}\right] dn$$

As in the case of four independent observations, and reducing to computer size, let

$$K = \sigma_m^{\,2}/R$$

$$x = n/R$$

$$\beta = \sigma_n/\sigma_m$$

The Fehlerintegral format becomes

$$\Phi\left[\frac{R}{\sqrt{3/2\,K}\,R}\left(x\cos\theta \pm \sqrt{1-x^2}\right)\sin\theta - \frac{x}{2\sqrt{3/2\,K}\,\beta}\right] \tag{D-17}$$

$$\frac{1}{2\sqrt{2\pi}\,\sigma_n} \quad \text{becomes} \quad \frac{1}{2\beta R\sqrt{2\pi K}}$$

$$\exp\left[-\frac{n^2}{2\,\sigma_n^{\,2}}\right] \text{becomes} \quad \exp\left[-\frac{x^2}{2K\beta^2}\right]$$

and

$$dn = R\,dx$$

The whole expression is then

$$\alpha = \frac{1}{2\beta\sqrt{2\pi K}} \int_{-1}^{+1} \left\{ \Phi\left[\frac{1}{\sqrt{3/2\,K}}\left(x\cos\theta + \sqrt{1-x^2}\sin\theta\right) - \frac{x}{2\sqrt{3/2\,K}\,\beta}\right] \right.$$
$$\left. - \Phi\left[\frac{1}{\sqrt{3/2\,K}}\left(x\cos\theta - \sqrt{1-x^2}\sin\theta\right) - \frac{x}{2\sqrt{3/2\,K}\,\beta}\right]\right\} \exp\left[-\frac{x^2}{2K\beta^2}\right] dx \tag{D-18}$$

To consider the symmetry, had we reversed the order of solution, using

$$n_1 = f_1(m) = m\cos\theta_x - \sqrt{R^2-m^2}\sin\theta_x$$

$$n_2 = f_2(m) = m\cos\theta_x + \sqrt{R^2-m^2}\sin\theta_x$$

and held v constant while solving for u, a direct reversal of m for n and n for m would have resulted as

$$\alpha = \frac{1}{2\sqrt{2\pi}\,\sigma_m} \int_{-R}^{R} \left\{ \Phi\left[\frac{1}{\sqrt{3/2}\,\sigma_n}\left(m\cos\theta + \sqrt{R^2-m^2}\,\sin\theta\right) - \frac{m}{2\sqrt{3/2}\,\sigma_m}\right] - \Phi\left[\frac{1}{\sqrt{3/2}\,\sigma_n}\left(m\cos\theta - \sqrt{R^2-m^2}\,\sin\theta\right) - \frac{m}{2\sqrt{3/2}\,\sigma_m}\right]\right\} \exp\left[-\frac{m^2}{2\,\sigma_m^2}\right] dm \tag{D-19}$$

If

$$K = \sigma_n^2/R$$

$$x = m/R$$

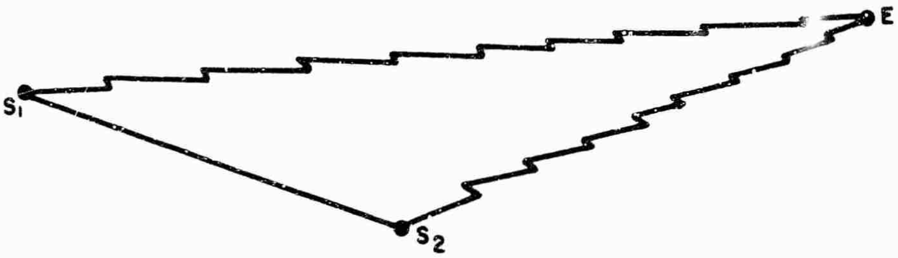
$$\beta = \sigma_m/\sigma_n$$

the final result would be precisely the same as Equation (D-18). Thus this solution demonstrates the same symmetrical aspect as the solution for four independent observers; values of α need only be calculated for $\beta > 1$.

APPENDIX E

PROBABILITY DISTRIBUTION OF TIMING ERRORS FOR A PAIR OF STATION OBSERVATIONS

The problem is to find the joint probability effect of a pair of time observations where each observation is independent of the other and each has a normal distribution of error.



The two stations S_1 and S_2 are measuring the time of arrival of the signature of event, E. Assume the measurements are t_1 and t_2 at S_1 and S_2 respectively. Assume the difference between these two readings is

$$x = t_2 - t_1 \qquad \text{(E-1)}$$

The probability distributions can then be expressed in terms of

$$p(t_2) \text{ and } p(t_2 - x)$$

We wish to find the most probable value of x and, more generally, the probability distribution of x with respect to t . These two events are assumed to be entirely independent of each other. Hence we are concerned with the product

$$p(t_2)p(t_2 - x) \qquad \text{(E-2)}$$

Since errors in measurements can be expressed as

$$t_1 = T_1 + \Delta t_1 \qquad \text{(E-3)}$$

$$t_2 = T_2 + \Delta t_2 \qquad \text{(E-4)}$$

where T_1 and T_2 are the exact values at S_1 and S_2 respectively, t_1 and $(t_1 - x)$ can take on all values within their error spectrum.

Since we are dealing with the subtraction of those time readings, what is the probability of subtracting a particular t_1 (t_{1i}) from a particular t_2 (t_{2i}) resulting in

$$x_j = t_{2i} - t_{1i}$$

The product of the probabilities of occurrence of t_{2i} and t_{1i}, namely, $p(t_{2i})\,p(t_{1i})$, or, $p(x_j) = p(t_{2i}) \times p(t_{2i} - x_j)$ is the answer. But there are many possibilities or combinations or readings t_{2i} and t_{1i} which will give this same exact x_j. Since the occurrence of any one combination excludes the others, the total of all possibilities and total probabilities is a summation process and the total

$$p(x_j) = \sum_i p\,(t_{2i})\,p\,(t_{2i} - x_j) \tag{E-5}$$

where i takes on all possible values that will result in a particular time difference, x_j.

Using a well-known theorem of probability, this can be generalized for all i by the expression

$$p(x) = \int_{-\infty}^{\infty} p\,(t_2)\,p\,(t_2 - x)\,dt_2 \tag{E-6}$$

To define the individual station probability densities let

$$p\,(t_2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t_2 - T_2)^2}{2\,\sigma^2}\right] \tag{E-7}$$

$$p\,(t_2 - x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t_2 - x - T_1)^2}{2\,\sigma^2}\right] \tag{E-8}$$

It is assumed there is no reason why instrumentation, human engineering, etc., should result in σ differing from one station to the next.

Then

$$p(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\,\sigma^2} \exp\left[-\frac{(t_2 - T_2)^2 - (t_2 - x - T_1)^2}{2\,\sigma^2}\right] dt_2 \tag{E-9}$$

To evaluate this integral let

$$a = T_2$$

$$b = T_1 + x$$

Then

$$p(x) = \frac{1}{2\pi\,\sigma^2} \int_{-\infty}^{\infty} \exp\left[-\frac{(t_2 - a)^2 - (t_2 - b)^2}{2\,\sigma^2}\right] dt_2 \tag{E-10}$$

To simplify the exponent (and drop the subscript),

$$\frac{-(t-a)^2 - (t-b)^2}{2\sigma^2} = \frac{-t^2 + 2at - a^2 - t^2 + 2bt - b^2}{2\sigma^2}$$

$$= \frac{-2t^2 + 2t(a+b) - (a^2 + b^2)}{2\sigma^2}$$

$$= -\frac{1}{\sigma^2}t^2 + \frac{a+b}{\sigma^2}t - \frac{a^2+b^2}{2\sigma^2}$$

$$= -\left(\frac{1}{\sigma^2}t^2 - \frac{a+b}{\sigma^2}t + \frac{a^2+b^2}{2\sigma^2}\right)$$

and

$$p(x) = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{\sigma^2}t^2 - \frac{a+b}{\sigma^2}t + \frac{a^2+b^2}{2\sigma^2}\right)\right] dt \qquad \text{(E-11)}$$

This can be evaluated from the integral form

$$\int_{-\infty}^{\infty} \exp\left[-(Ax^2 + 2Bx + C)\right] dx = \sqrt{\frac{\pi}{A}}\exp\left[\frac{B^2 - AC}{A}\right] \quad [A > 0] \qquad \text{(E-12)}$$

Let

$$A = \frac{1}{\sigma^2} \qquad [\text{hence } A \text{ is} > 0]$$

$$B = -\frac{a+b}{2\sigma^2}$$

$$C = \frac{a^2+b^2}{2\sigma^2}$$

Then the integral equation (E-12) becomes

$$\sqrt{\frac{\pi}{1/\sigma^2}}\exp\left[\frac{\left(\frac{a+b}{2\sigma^2}\right)^2 - \frac{1}{\sigma^2}\,\frac{a^2+b^2}{2\sigma^2}}{1/\sigma^2}\right]$$

To simplify the exponent,

$$\frac{\left(\frac{a+b}{2\sigma}\right)^2 - \frac{a^2+b^2}{2\sigma^4}}{1/\sigma^2} = \sigma^2\left(\frac{a^2+2ab+b^2}{4\sigma^4} - \frac{a^2+b^2}{2\sigma^4}\right)$$

$$= \frac{1}{4\sigma^2}\left(a^2 + 2ab + b^2 - 2a^2 - 2b^2\right)$$

$$= -\frac{1}{4\sigma^2}\left(a^2 - 2ab + b^2\right)$$

$$= -\left(\frac{a-b}{2\sigma}\right)^2$$

and the integral equation (E-13) becomes

$$\sigma\sqrt{\pi}\ \exp\left[-\left(\frac{a-b}{2\sigma}\right)^2\right] ,$$

then,

$$p(x) = \frac{1}{2\pi\sigma^2} \times \sigma\sqrt{\pi}\left[-\left(\frac{a-b}{2\sigma}\right)^2\right] \qquad \text{(E-13)}$$

Replacing a and b gives,

$$p(x) = \frac{1}{2\sigma\sqrt{\pi}}\ \exp\left[-\frac{(T_2 - T_1 - x)^2}{4\sigma^2}\right] \qquad \text{(E-14)}$$

If we now define terms such that

$$M = T_2 - T_1 \qquad \text{(E-15)}$$

$$\sigma_x^{\ 2} = 2\sigma^2 \qquad \text{(E-16)}$$

$$p(x) = \frac{1}{\sigma_x\sqrt{2\pi}}\ \exp\left[-\frac{(M-x)^2}{2\sigma_x^{\ 2}}\right] = p(t_2 - t_1) \qquad \text{(E-17)}$$

therefore, the most probable value of x is $x = M$ for which p(x) is given at

$$M = T_2 - T_1 \tag{E-18}$$

with a standard deviation

$$\sigma_x = \sqrt{2}\,\sigma$$

This function is depicted in Figure E-1 below. Note that while the symbol p(x) is used, the function (E-17) is a probability density function for which the current trend is to use the notation f(x).

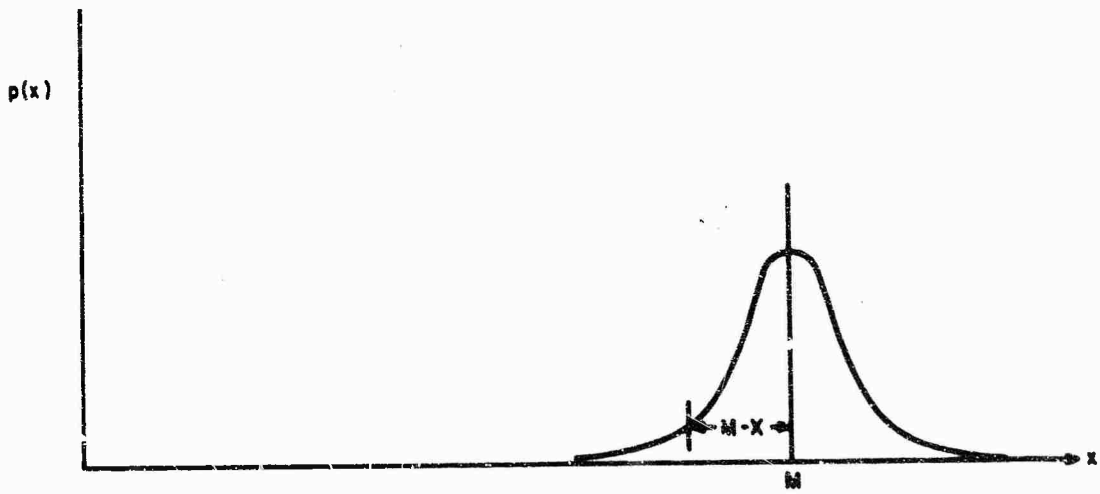


Figure E-1. Graphical Results of p(x)

APPENDIX F

EEP (ELLIPTICAL ERROR PROBABILITY) CALCULATIONS FOR A NORMAL ERROR PROBABILITY DISTRIBUTION FOR FOUR OBSERVERS

As stated in the text, there is nothing inherently natural in this type of system about a circular area of probable occurrence; and, as further stated, if we can break free of QCEP compatibility requirements, we can enjoy distinct advantages in inherent elliptical contours of error.

To determine what the inherent contours really are, we shall pursue the educated guess that it is possible to determine the probability of error directly from the time domain.

As shown in Appendix E,

$$p(x) = \frac{1}{\sigma_x 2\pi} \exp\left[- \frac{(M - x)^2}{2\,\sigma_x^2} \right] \qquad \text{(F-1)}$$

For the pair of stations whose timing error would result in deviations or displacements in n, it is convenient to first let

$$\Delta T_n = M - x \qquad \text{(F-2)}$$

and

$$p(\Delta T_n) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[- \frac{\Delta T_n^{\,2}}{2\,\sigma_x^2} \right] \qquad \text{(F-3)}$$

Likewise, displacements in m would result from

$$p(\Delta T_m) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[- \frac{\Delta T_m^{\,2}}{2\,\sigma_x^2} \right] \qquad \text{(F-4)}$$

Recall from the text that

$$\alpha = \int\int_A p(m)\, p(n)\, dm\, dn \qquad \text{(F-5)}$$

Since it was also shown in the text that

$$n = \Gamma_n \Delta t_n \qquad \text{(F-6)}$$

$$m = \Gamma_m \Delta t_m \qquad \text{(F-7)}$$

it can be stated from the principal

$$p\,(m) = p(\Delta t_m)\,\frac{d\Delta t_m}{dm} \tag{F-8}$$

that

$$p\,(m)\,dm = p(\Delta t_m)\,d\Delta t_m \tag{F-9}$$

$$p\,(n)\,dn = p(\Delta t_n)\,d\Delta t_n \tag{F-10}$$

Hence

$$\alpha = \iint_A p(m)\,p(n)\,dm\,dn = \iint_T p(\Delta t_m)\,p(\Delta t_n)\,d\Delta t_m\,d\Delta t_n \tag{F-11}$$

but substituting Equations (F-3) and (F-4)

$$\alpha = \frac{1}{\sigma_x^2\,2\pi}\iint_T \exp\left[-\frac{(\Delta t_n^2 + \Delta t_m^2)}{2\,\sigma_x^2}\right] d\Delta t_m\,d\Delta t_n \tag{F-12}$$

Δt_m and Δt_n are independent of each other and can represent variables in the usual orthogonal relationship for purposes of geometric analysis.

Consider Figure F-1 and the results of letting the magnitude of timing errors from 0 to Δt_m and 0 to Δt_n be represented by

$$\Delta t_m^2 + \Delta t_n^2 = S^2 \tag{F-13}$$

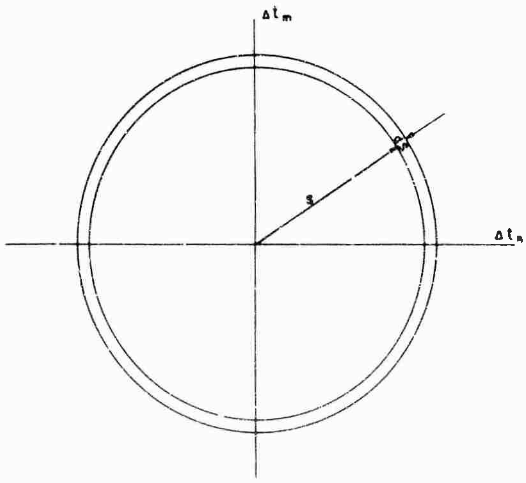


Figure F-1. An Element of EEP Analysis in the Time Domain (Four Observers)

This will convert Equation (F-12) to

$$\alpha = \frac{1}{2\pi \sigma_x^2} \iint_A \exp\left[-\frac{S^2}{2\sigma_x^2}\right] dA \qquad \text{(F-14)}$$

where A is bounded by the circle of radius S.

The element of area can be considered the ring (for the first integration)

$$dA = 2\pi\ s\ d\ s$$

whence

$$\alpha = \frac{1}{2\pi \sigma_x^2} \int_0^S \exp\left[-\frac{s^2}{2\sigma_x^2}\right] 2\pi\ s\ d\ s$$

$$= - \int_0^S \exp\left[-\frac{s^2}{2\sigma_x^2}\right] d\left(-\frac{s^2}{2\sigma_x^2}\right)$$

$$\alpha = -\left[\exp\left(-\frac{s^2}{2\sigma_x^2}\right)\right]_0^S = -\left[\exp\left(-\frac{s^2}{2\sigma_x^2}\right) - 1\right]$$

$$\alpha = 1 - \exp\left[-\frac{S^2}{2\sigma_x^2}\right] \qquad \text{(F-15)}$$

This is the probability of a fix lying within the area resulting from the probability of error measurements of Δt_n and Δt_m in the time domain and LOP errors of n and m in the space domain. To pursue this further α is also related to the probability of making LOP errors of

$$n = \Gamma_n \Delta t_n$$

$$m = \Gamma_m \Delta t_m$$

Recalling from Equation (C-1) Appendix C, that

$$\alpha = \frac{1}{2\pi \sigma_n \sigma_m} \iint_A \exp - \left[\frac{n^2}{2\sigma_n^2} + \frac{m^2}{2\sigma_m^2}\right] dm dn \qquad \text{(F- 16)}$$

and that

$$\sigma_n = \sigma_x \Gamma_n$$

$$\sigma_m = \sigma_x \Gamma_m$$

then

$$\alpha = \frac{1}{2\pi \Gamma_n \Gamma_m \sigma_x^2} \int\int_A \exp\left[-\frac{1}{\sigma_x^2}\left(\frac{n^2}{2\Gamma_n^2} + \frac{m^2}{2\Gamma_m^2}\right)\right] dm\,dn \qquad \text{(F-17)}$$

since

$$\frac{n^2}{2\Gamma_n^2} + \frac{m^2}{2\Gamma_m^2} = \frac{\Delta T_m^2}{2} + \frac{\Delta T_n^2}{2} = \frac{S^2}{2} \qquad \text{(F-18)}$$

We have established that taking the points Δt_n and Δt_m from the circle S in the time domain results in an elliptical figure in the space domain which is characterized by the quantity

$$\frac{1}{\sigma_x^2}\left(\frac{n^2}{2\Gamma_n^2} + \frac{m^2}{2\Gamma_m^2}\right) \quad \text{or} \quad \frac{n^2}{2\sigma_n^2} + \frac{m^2}{2\sigma_m^2} = \frac{1}{\sigma_x^2} \quad \frac{S^2}{2}$$

It must be remembered that α is the probability of making any combination of timing errors, Δt_n and Δt_m, up to and including S. This results in a spatial ellipse with the same probability of a fix being anywhere within or on this ellipse. The fact that n and m are oblique has had nothing to do with the analysis thus far. By letting

$$n_{Max} = \Gamma_n S \qquad \text{(F-19)}$$

$$m_{Max} = \Gamma_m S \qquad \text{(F-20)}$$

the probability α associated with any ellipse

$$\frac{n_{Max}^2}{2\sigma_n^2} + \frac{m_{Max}^2}{2\sigma_m^2} = 1$$

can be determined. However, in plotting this ellipse, n and m must be the true non-orthogonal axes. Sample curves are presented in Figure F-2.

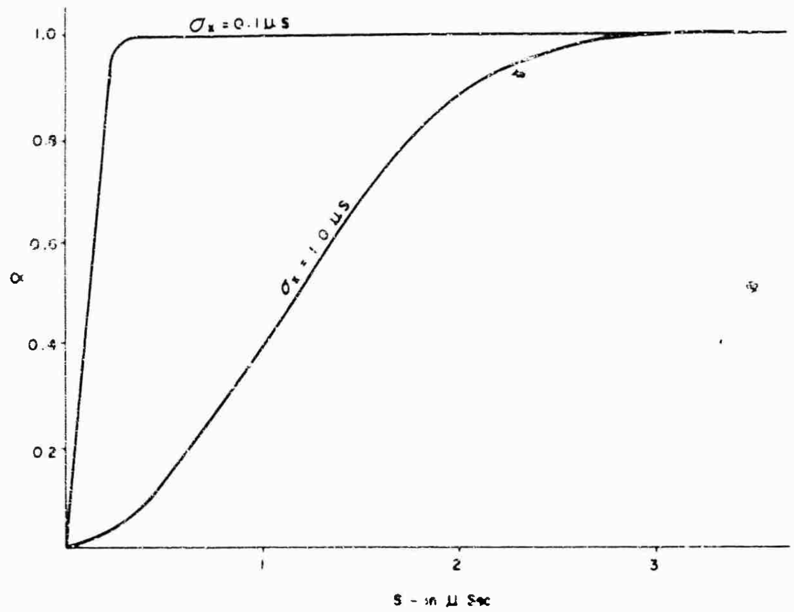


Figure F-2. Sample Curves of $1 - \exp\left[-\frac{S^2}{2\sigma_x^2}\right]$

Thus far we have not placed any constraints or special significance on the quantity S itself, other than in Equation (F-18). But this represents a specific -- one might say natural -- ellipse. Can we make concentric expansions and contractions to this ellipse? To answer, it may help to introduce the idea of S_0 for this "natural" ellipse such that

$$S_o = \sqrt{2}\,\sigma_x \qquad \text{(F-21)}$$

representing that limit of $\sqrt{\Delta t_n^2 + \Delta t_m^2}$ which is also the limit, or boundary, of the ellipse

$$\frac{n^2}{2\sigma_n^2} + \frac{m^2}{2\sigma_m^2} = 1$$

To change to a different ellipse of

$$\frac{n^2}{\lambda^2\, 2\sigma_n^2} + \frac{m^2}{\lambda^2\, 2\sigma_m^2} = 1$$

equivalent to the time domain change of

$$S = \lambda S_o \qquad \text{(F-22)}$$

and introducing this idea into α gives

$$\alpha = 1 - \exp\left[-\frac{\lambda^2 S_o^2}{2\sigma_x^2}\right] = \frac{1}{2\pi\sigma_n\sigma_m}\iint_A \exp\left[-\frac{1}{\lambda^2}\left(\frac{n^2}{2\sigma_n^2}+\frac{m^2}{2\sigma_m^2}\right)\right]dmdn \tag{F-23}$$

but for the "natural" ellipse $\lambda = 1$ and

$$S_o^2 = 2\sigma_x^2 \tag{F-24}$$

$$\alpha = 1-e^{-1} = 0.6322$$

resulting in the general formula

$$\alpha = 1-e^{-\lambda^2} \tag{F-25}$$

with the spatial elliptical relations

$$\frac{n^2}{a^2}+\frac{m^2}{b^2} = 1 \tag{F-26}$$

where

$$a = \lambda a_o = \lambda\sqrt{2}\,\sigma_n = \lambda\sqrt{2}\,\sigma_x\,\Gamma_n \tag{F-27}$$

$$b = \lambda b_o = \lambda\sqrt{2}\,\sigma_m = \lambda\sqrt{2}\,\sigma_x\,\Gamma_m \tag{F-28}$$

$$a/b \text{ or } b/a = \Gamma_n/\Gamma_m \text{ or } \Gamma_m/\Gamma_n = \beta \quad [\beta > 1] \tag{F-29}$$

Again note that these are pseudo- or orthogonal-elliptical quantities which must be converted in accordance with the disciplines discussed in Appendix H to get the true spatial picture.

APPENDIX G

EEP CALCULATIONS FOR A NORMAL ERROR PROBABILITY DISTRIBUTION FOR THREE OBSERVERS

To determine what the inherent elliptical contours of error are, we shall pursue the determination of error directly in the time domain. For the pair of stations whose timing error would result in deviations or displacements in n it was shown in Appendix E that

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{(M-x)^2}{2\sigma_x^2}\right] \tag{G-1}$$

Further, for this pair of stations let

$$\Delta t_n = M-x = \Delta t_B + \Delta t_C \tag{G-2}$$

as a simpler representation of the net error for the time difference of a particular pair of stations. Then

$$f(\Delta t_n) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{\Delta t_n^2}{2\sigma_x^2}\right] \tag{G-3}$$

Likewise, displacements in m would result from Δt_m with probability density

$$f(\Delta t_m) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{\Delta t_m^2}{2\sigma_x^2}\right] \tag{G-4}$$

where

$$\Delta t_m = M-y = \Delta t_A + \Delta t_B \tag{G-5}$$

As shown in Appendix J, the joint probability density function is given by,

$$f(\Delta t_n, \Delta t_m) = \frac{1}{\sqrt{3}\pi\sigma_x^2} \exp\left[-\frac{2/3(\Delta t_n^2 + \Delta t_m^2 - \Delta t_n \Delta t_m}{\sigma_x^2}\right] \tag{G-6}$$

and the summation of points within the ellipse

$$\frac{\Delta t_n^2}{(3/2)\sigma_x^2} + \frac{\Delta t_m^2}{(3/2)\sigma_x^2} - \frac{\Delta t_m \Delta t_n}{(3/2)\sigma_x^2} = 1 \tag{G-7}$$

gives a probability of joint error for

$$\left.\begin{array}{l} 0 \leq \Delta t_n \leq \Delta t_N \\ 0 \leq \Delta t_m \leq \Delta t_M \end{array}\right\} t \qquad \text{(G-8)}$$

which is given by

$$p(\Delta t_n, \Delta t_m) = \alpha = \frac{1}{\sqrt{3}\,\pi\,\sigma_x^2} \int\int_t \exp - \left[\frac{\Delta t_n^{\,2}}{(3/2)\sigma_x^2} + \frac{\Delta t_m^{\,2}}{(3/2)\,\sigma_x^2} - \frac{\Delta t_m \Delta t_n}{(3/2)\sigma_x^2} \right] d\Delta t_n d\Delta t_m \qquad \text{(G-9)}$$

To evaluate this integral refer to Figure G-1 which may be considered a sketch of Equation (G-7). It can be shown algebraically that this is equivalent to a skewed ellipse onto axes $\Delta t_n'$ and $\Delta t_m'$ whose equation is given by

$$\frac{(\Delta t_n')^2}{a^2} + \frac{(\Delta t_m')^2}{b^2} = 1 \qquad \text{(G-10)}$$

The major and minor axes a and b can be determined by

$$\frac{1}{a^2} = \frac{\cos^2 \phi}{(3/2)\sigma_x^2} + \frac{\sin^2 \phi}{(3/2)\sigma_x^2} - \frac{\sin \phi \cos \phi}{(3/2)\sigma_x^2}$$

$$\frac{1}{a^2} = \frac{1 - \sin \phi \cos \phi}{(3/2)\sigma_x^2} \qquad \text{(G-11)}$$

$$a^2 = \frac{(3/2)\,\sigma_x^2}{1 - \sin \phi \cos \phi}$$

Similarly,

$$b^2 = \frac{(3/2)\sigma_x^2}{1 + \sin \phi \cos \phi} \qquad \text{(G-12)}$$

ϕ is determined by

$$\begin{aligned} \phi &= \text{arc tan} \left[\left(\frac{3}{2}\,\sigma_x^2 - \frac{3}{2}\,\sigma_x^2 \right) \pm \sqrt{\left(\frac{3}{2}\,\sigma_x^2 - \frac{3}{2}\,\sigma_x^2 \right) + 1} \right] \\ &= \text{arc tan } 1 \\ &= 45^\circ \end{aligned}$$

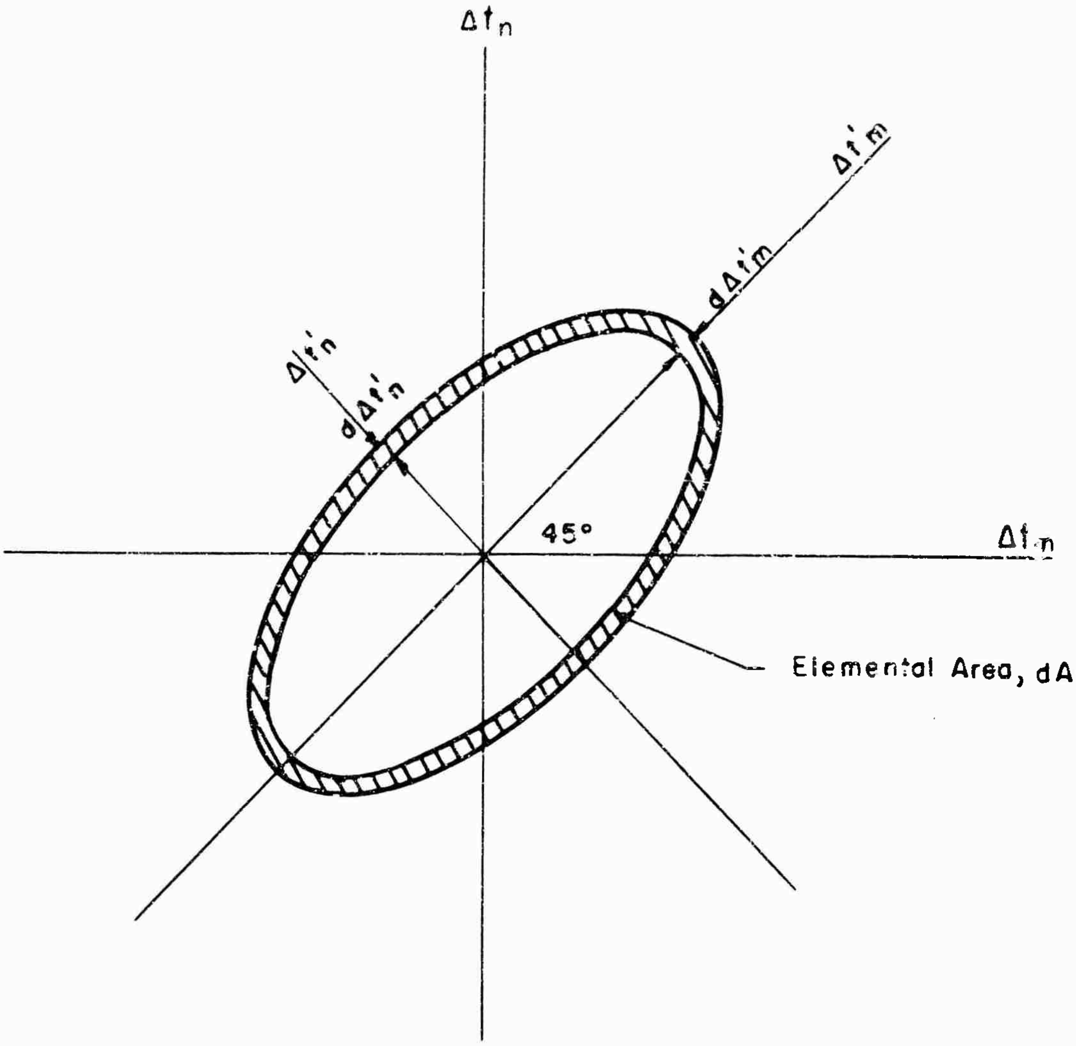


Figure G-1. An Element of EEP Analysis in the Time Domain
(Three Observers)

Thus

$$a^2 = \frac{\frac{3}{2}\sigma_x^2}{1 - \frac{1}{2}} = 3\,\sigma_x^2$$

$$b^2 = \frac{\frac{3}{2}\sigma_x^2}{1 + \frac{1}{2}} = \sigma_x^2$$

Returning to Equation (G-10), the skewed ellipse becomes

$$\frac{(\Delta t_n')^2}{3\sigma_x^2} + \frac{(\Delta t_m')^2}{\sigma_x^2} = 1 \qquad \text{(G-13)}$$

Using π ab for the area of an ellipse, the elemental area dA of the skewed ellipse on axes $\Delta t_n'$, $\Delta t_m'$ is determined by

$$\begin{aligned} dA &= \pi\, \Delta t_N'\, \Delta t_M' - \pi (\Delta t_N' - d\Delta t_N')(\Delta t_M' - d\Delta t_M') \\ dA &= \pi\, (\Delta t_N'\, d\Delta t_m' + \Delta t_M'\, d\Delta t_n' - d\Delta t_n'\, d\Delta t_m') \end{aligned} \qquad \text{(G-14)}$$

To reduce the integration from a double to a single summation, we evaluate, in effect,

$$\sigma = \frac{1}{\sqrt{3}\,\pi\, \sigma_x^2} \iint_{t'} \exp - \left[\frac{(\Delta t_n')^2}{3\,\sigma_x^2} + \frac{(\Delta t_m')^2}{\sigma_x^2} \right] d\Delta t_n'\, d\Delta t_m' \qquad \text{(G-15)}$$

for

$$\begin{aligned} 0 &\leq \Delta t_n' \leq \Delta t_N' \\ 0 &\leq \Delta t_m' \leq \Delta t_M' \end{aligned} \qquad \text{(G-16)}$$

The exponent of e is the constant value of any point on the elliptical ring dA as $d\Delta t_n'$ and $d\Delta t_m'$ approach zero. But the total area, and its associated joint density function, can be considered a concentric series of such rings; as the maxima $\Delta t_N'$ and $\Delta t_M'$ move from zero to A and B. For example, we have the relationship

$$\frac{\Delta t_N'}{\Delta t_M'} = \frac{A}{B} \qquad \text{(G-17)}$$

This permits us to generalize Equation (G-14)

$$dA = \pi \left(\frac{A}{B}\, \Delta t_M'\, d\Delta t_n' + \frac{B}{A}\, \Delta t_N'\, d\Delta t_n' - d\Delta t_n'\, d\Delta t_m'\right) \qquad \text{(G-18)}$$

But from Equation (G-13)

$$\frac{A}{B} = \sqrt{3}\,, \quad \frac{B}{A} = \frac{1}{\sqrt{3}}$$

giving

$$dA = \pi \left(\sqrt{3}\, \Delta t_M' \, d\Delta t_m' + \frac{1}{\sqrt{3}} \Delta t_N' \, d\Delta t_n' - d\Delta t_n' \, d\Delta t_m' \right) \tag{G-19}$$

The integral has now become

$$\alpha = \frac{1}{\sqrt{3}\, \pi \sigma_x^2} \int_A \exp - \left[\frac{(\Delta t_n')^2}{3\sigma_x^2} + \frac{(\Delta t_m')^2}{\sigma_x^2} \right] dA$$

$$\alpha = \frac{\pi}{\sqrt{3}\, \pi \sigma_x^2} \int_A \exp - \left[\frac{(\Delta t_n')^2}{3\sigma_x^2} + \frac{(\Delta t_m')^2}{\sigma_x^2} \right] \times$$

$$\times \left(\sqrt{3}\, \Delta t_M' \, d\Delta t_m' + \frac{1}{\sqrt{3}} \Delta t_N' \, d\Delta t_m' - d\Delta t_m' \, d\Delta t_n' \right) \tag{G-20}$$

Let

$$\frac{(\Delta t_n')^2}{3\sigma_x^2} + \frac{(\Delta t_m')^2}{\sigma_x^2} = S^2 \tag{G-21}$$

be a general ellipse whose value is constant for any given ellipse. Since it will not matter whereon any given ellipse an evaluation is performed, i.e., $\Delta t_n' = \Delta t_N'$, $\Delta t_m' = 0$, and $\Delta t_n' = 0$, $\Delta t_m' = \Delta t_M'$ are valid maximum points on the ellipse S and we may write

$$\alpha = \frac{1}{\sqrt{3}\sigma_x^2} \left\{ \int \exp \left[- \frac{(\Delta t_N')^2}{3\sigma_x^2} \right] \times \frac{\Delta t_N'}{\sqrt{3}} d\Delta t_n' + \int \sqrt{3} \exp \left[- \frac{(\Delta t_M')^2 \text{ Max}}{\sigma_x^2} \right] \Delta t_M' \, d\Delta t_m \right.$$

$$\left. - \int \exp[-S^2] d\Delta t_n' \, d\Delta t_m' \right\} \tag{G-22}$$

The last term may also be written

$$\left\{ \int \exp \left[- \frac{(\Delta t_M')^2}{\sigma_x^2} \right] d\Delta t_m' \right\} d\Delta t_n'$$

It is obvious that the expression in the bracket is a finite integral. Hence in the limit

$$\lim_{d\Delta t_N' \to 0} \left\{ \int \exp - \left[\frac{(\Delta t_M')^2}{\sigma_x^2} \right] d\Delta t_m' \right\} d\Delta t_n' = 0 \tag{G-23}$$

the evaluation of α proceeds as follows

$$\alpha = -\frac{1}{\sqrt{3}} \left\{ \int_0^{a'} \frac{3}{2\sqrt{3}} \exp\left[-\frac{(\Delta t_N')^2_{Max}}{3\,\sigma_x^2}\right] d\left(-\frac{\Delta t_n'^2}{3\,\sigma_x^2}\right) \right.$$

$$\left. -\frac{1}{\sqrt{3}} \int_0^{b'} \frac{\sqrt{3}}{2} \exp\left[-\frac{(\Delta t_M')^2_{Max}}{\sigma_x^2}\right] d\left(-\frac{\Delta t_m'}{\sigma_x^2}\right) \right\}$$

(G-24)

The limits a' and b' of $\Delta t_M'$ and $\Delta t_N'$ are discussed later. Then

$$\alpha = -\frac{1}{2} \exp\left(-\frac{(\Delta t_N')^2_{Max}}{3\,\sigma_x^2}\right)\Bigg]_0^{a'} - \frac{1}{2} \exp\left(-\frac{(\Delta t_M')^2_{Max}}{\sigma_x^2}\right)\Bigg]_0^{b'} \qquad \text{(G-25)}$$

$$\alpha = \frac{1}{2}\left(1-\exp\left[-\frac{-(a')^2}{3\sigma_x^2}\right]\right) + \frac{1}{2}\left(1-\exp\left[-\frac{(b')^2}{\sigma_x^2}\right]\right)$$

$$\alpha = 1 - \frac{1}{2}\left(\exp\left[-\frac{(a')^2}{3\sigma_x^2}\right] + \exp\left[-\frac{(b')^2}{\sigma_x^2}\right]\right) \qquad \text{(G-26)}$$

It is interesting to note that for special cases of a zero ellipse, and an all-inclusive infinite ellipse, the limits and probability would be

$$a' = b' = 0, \ \alpha = 0$$

$$a' = b' = \infty, \ \alpha = 1$$

which checks our knowledge of boundary conditions.

Since, as has been pointed out for the joint probability density functions of this system,

$$(a')^2 = 3(b')^2$$

we have finally

$$\alpha = 1 - \exp\left[-\frac{(a')^2}{\sigma_x^2}\right] = 1 - \exp\left[-\frac{(b')^2}{3\,\sigma_x^2}\right] \qquad \text{(G-27)}$$

or, more generally, for any point on the general ellipse S where

$$S^2 = \frac{(\Delta t_n')^2}{3\sigma_x^2} + \frac{(\Delta t_m')^2}{\sigma_x^2} = \frac{\Delta t_n^2}{3/2\,\sigma_x^2} + \frac{\Delta t_m^2}{3/2\,\sigma_x^2} - \frac{\Delta t_m \Delta t_n}{3/2\,\sigma_x^2} \tag{G-28}$$

$$\alpha = 1 - e^{-S^2}$$

and the points a' and b' are the special "max" values of $\Delta t_N'$ and $\Delta t_M'$ which give the values of S.

When S is unity we have the natural standard deviation ellipse. The probability that the position fix lies within or on this ellipse is then

$$\alpha = 1 - e^{-1} = 63.22\%$$

For concentric ellipses for other values of S, the elliptical equation can be considered as

$$1 = \frac{(\Delta t_n'')^2}{S^2\,3\sigma_x^2} + \frac{(\Delta t_m'')^2}{S^2\,\sigma_x^2} = \frac{\Delta t_n^2}{S^2\,3/2\,\sigma_x^2} + \frac{\Delta t_m^2}{S^2\,3/2\,\sigma_x^2} - \frac{\Delta t_m \Delta t_n}{S^2\,3/2\,\sigma_x^2} \tag{G-29}$$

where

$$\Delta t_n'' = S\Delta t_n'$$

$$\Delta t_m'' = S\Delta t_m'$$

and the unprimed Δt_n, Δt_m are also changed by the factor S . Thus we may consider any size S ellipse we wish.

To relate this situation to the space domain it is recalled that

$$n = \Delta t_n \Gamma_n$$

$$m = \Delta t_m \Gamma_m$$

$$\sigma_n = \Gamma_n \sigma_x$$

$$\sigma_m = \Gamma_m \sigma_x$$

Utilizing these quantities in Equation (G-29) gives the result

$$\frac{n^2}{3/2\,\sigma_n^2} + \frac{m^2}{3/2\,\sigma_m^2} - \frac{mn}{3/2\,\sigma_m \sigma_n} = S^2 \tag{G-30}$$

The values of a and b which were used in Appendix I to derive the transforms are then given by

$$a = S\sqrt{3/2}\,\sigma_n$$

$$b = S\sqrt{3/2}\,\sigma_m$$

These are pseudo- or orthogonal-elliptical quantities which must be transformed in accordance with Appendix I to get the true spatial picture.

APPENDIX H

DERIVATION OF THE TRUE ELLIPSE TRANSFORMS FOR FOUR OBSERVERS

The general problem is to derive an expression for the true ellipse in terms of its oblique axes such that a complete picture of the spatial probability contours is obtained.

It is shown in Appendix A that the value of the distance R from a fix point P to any displaced point P' is given by

$$R^2 = \frac{1}{\sin^2 \theta_x} (m^2 + n^2 - 2\, mn \cos \theta_x) \tag{H-1}$$

where

θ_x = crossing angle of the LOP's

m = error or displacement of the LOP for the station pair AB measured along the line m at right angle to r_m.

n = error displacement of the LOP for the station pair BC measured along the line n at right angles to r_n.

It is also shown in Appendix F, Equation (F-26), that there exists an elliptical mathematical relation of a constant contour of probability such that

$$\frac{n^2}{a^2} + \frac{m^2}{b^2} = 1 \tag{H-2}$$

The question is: "When the values of n and m which satisfy this equation are substituted into equation H-1, can we obtain the "max" and "min" of the resulting R, and what are these values?"

To answer the first part of the question, the variations of R are studied in an example.

First m is replaced in Equation (H-2) by

$$m = b\sqrt{1 - \frac{n^2}{a^2}} \tag{H-3}$$

giving

$$R^2 = \frac{1}{\sin^2 \theta_x} \left[n^2 + b^2 \left(1 - \frac{n^2}{a^2}\right) - 2nb\sqrt{1 - \frac{n^2}{a^2}} \cos \theta_x \right] \tag{H-4}$$

$$R = \frac{1}{\sin\theta} \sqrt{n^2(1 - b^2/a^2) + b^2 - 2cn\sqrt{1 - n^2/a^2}\cos\theta} \qquad \text{(H-5)}$$

For the example chosen

$$\theta = 30^{\circ}$$
$$a^2 = 8$$
$$b^2 = 2$$

and

$$R = \sqrt{3n^2 - 9.8n\sqrt{1 - n^2/8} + 8}$$

The results of this calculation are shown in Figure H-1. This demonstrates that there is a value of $R = f(n)$ which in the usual orthogonal sense will contain a maximum and minimum.

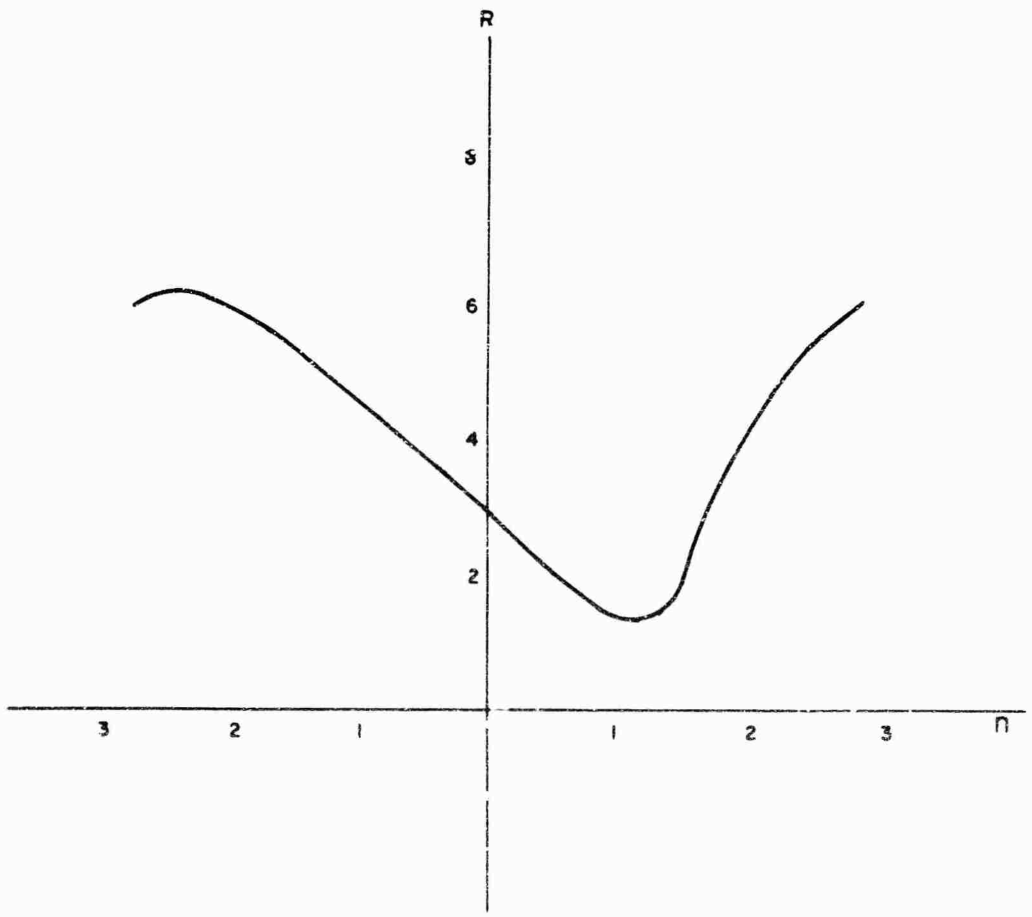


Figure H-1. A Plot of an Example of Calculating R to a Point on the (Oblique) Ellipse

To derive the general expression for all maximum and minimum, we return to Equation (H-4). By inspection of the ellipse of Equation (H-2) projected onto the oblique system n, m (as in Figure 6) it is seen that R_{max} and R_{min} are obtained when dR = 0. But since dn is independent of dR, if dR is zero

$$\frac{dR}{dn} = 0$$

From equation (H-5),

$$\frac{dR}{dn} = \frac{1}{\sin\theta} \times \frac{1}{2}\left[n^2 + b^2\left(1 - \frac{n^2}{b^2}\right) - 2nb\sqrt{1 - \frac{n^2}{b^2}}\cos\theta_x\right]^{-1/2}$$

$$\left[2n - 2\frac{b^2}{a^2}n - 2b\left(\frac{-n^2}{a^2\sqrt{1 - \frac{n^2}{a^2}}} + \sqrt{1 - \frac{n^2}{a^2}}\right)\cos\theta_x\right]$$

Equating to zero and solving for n^2 proceeds as follows

$$n - \frac{b^2}{a^2}n + \frac{b}{a}\frac{n^2}{\sqrt{1 - \frac{n^2}{a^2}}}\cos\theta - b\sqrt{1 - \frac{n^2}{a^2}}\cos\theta = 0$$

$$n\left(1 - \frac{b^2}{a^2}\right) + \frac{b_n^2}{a^2\sqrt{1 - \frac{n^2}{a^2}}}\cos\theta_x - b\sqrt{1 - \frac{n^2}{a^2}}\cos\theta = 0$$

$$n\sqrt{1 - \frac{n^2}{a^2}}\left(1 - \frac{b^2}{a^2}\right) + \frac{b}{a^2}n^2\cos\theta_x - b\left(1 - \frac{n^2}{a^2}\right)\cos\theta_x = 0$$

$$n\sqrt{1 - \frac{n^2}{a^2}}\left(1 - \frac{b^2}{a^2}\right) + 2\frac{b}{a^2}\cos\theta\, n^2 - b\cos\theta = 0$$

$$n\sqrt{1 - \frac{n^2}{a^2}}\left(1 - \frac{b^2}{a^2}\right) = b\cos\theta - 2\, b/a\cos\theta_n{}^2$$

Squaring both sides

$$n^2\left(1-\frac{n^2}{a^2}\right)\left(1-\frac{b^2}{a^2}\right) = b^2\cos^2\theta - 4\frac{b^2}{a^2}\cos^2\theta\, n^2 + 4\frac{b^2}{a^4}\cos^2\theta\, n^4$$

$$\left(n^2-\frac{n^4}{a^2}\right)\left(1-\frac{b^2}{a^2}\right) = 4\frac{b^2}{a^4}\cos^2\theta\, n^4 - 4\frac{b^2}{a^2}\cos^2\theta\, n^2 + b^2\cos^2\theta$$

$$n^2\left(1-\frac{b^2}{a^2}\right)^2 - \frac{n^4}{a^2}\left(1-\frac{b^2}{a^2}\right) = 4\frac{b^2}{a^4}\cos^2\theta\, n^4 - 4\frac{b^2}{a^2}\cos^2\theta\, n^2 + b^2\cos^2\theta$$

$$4\frac{b^2}{a^4}\cos^2\theta\, n^4 + \frac{1}{a^2}\left(1-\frac{b^2}{a^2}\right)n^4 - \left(1-\frac{b^2}{a^2}\right)n^2 - 4\frac{b^2}{a^2}\cos^2\theta\, n^2 + b^2\cos^2\theta = 0$$

$$\left[4\frac{b^2}{a^2}\cos^2\theta + \left(1-\frac{b^2}{a^2}\right)^2\right]\frac{n^4}{a^2} - \left[4\frac{b^2}{a^2}\cos^2\theta + \left(1-\frac{b^2}{a^2}\right)^2\right]n^2 + b^2\cos^2\theta = 0$$

$$\frac{1}{a^2}\left[4\frac{b^2}{a^2}\cos^2\theta + \left(1-\frac{b^2}{a^2}\right)\right]n^4 - \left[4\frac{b^2}{a^2}\cos^2\theta + \left(1-\frac{b^2}{a^2}\right)^2\right]n^2 + b^2\cos^2\theta = 0$$

Solving for n^2 using the binomial theorem

$$\frac{1}{a^2}n^4 - n^2 + \frac{b^2\cos^2\theta}{4\frac{b^2}{a^2}\cos^2\theta + \left(1-b^2/a^2\right)} = 0$$

$$n^2 = \frac{1 \pm \sqrt{1 - \dfrac{4\,b^2/a^2\cos^2\theta}{4b^2/a^2\cos^2\theta + (1-b^2/a^2)^2}}}{2/a^2}$$

$$\frac{n^2}{a^2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \dfrac{1}{1 + \dfrac{(1-b^2/a^2)^2}{4\,b^2/a^2\cos^2\theta}}} \qquad \text{(H-6)}$$

For computation purposes, let

$$x = \frac{1 - b^2/a^2}{2\, b/a \cos\theta} = \frac{a/b - b/a}{2 \cos\theta} \tag{H-7}$$

Then

$$\frac{n^2}{a^2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{1 + x^2}}\right) \tag{H-8}$$

but from Equation (H-2)

$$\frac{m^2}{b^2} = 1 - \frac{n^2}{a^2}$$

and

$$\frac{m^2}{b^2} = 1 - \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

$$\frac{m^2}{b^2} = \frac{1}{2}\left(1 \mp \sqrt{1 - \frac{1}{1 + x^2}}\right) \tag{H-9}$$

From Equations (H-8 and H-9), the values of m and n which give R_{max} and R_{min} are

$$n = \pm \frac{a}{\sqrt{2}} \sqrt{1 \pm \sqrt{1 - \frac{1}{1 + x^2}}} \tag{H-10}$$

$$m = \pm \frac{b}{\sqrt{2}} \sqrt{1 \mp \sqrt{1 - \frac{1}{1 + x^2}}} \tag{H-11}$$

Insertion of these quantities into Equation (H-1) presents a problem in the selection of polarity for the term mn cos θ_x. This has been solved by graphical analysis which shows that in the determination of R_{max}, m and n are always of opposite polarity regardless of whether the pseudo-ellipse is plotted with its major axis along m or n. Similarly, graphical analysis shows that for the point on the true ellipse which represents R_{min} m and n are of the same polarity.

Further analysis with regard to the selection of polarity shows that when $a > b$, R_{max} is obtained from the combination of

$$\frac{n^2}{a^2} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

$$\frac{m^2}{b^2} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

Conversely when $b > a$, R_{max} is obtained from the combination of

$$\frac{n^2}{a^2} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

$$\frac{m^2}{b^2} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

Thus the formulas for the true major and minor axes become:

For $a > b$

$$R^2_{max} = \frac{1}{2 \sin^2 \theta}\left\{ a^2 \left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right) + b^2 \left(1 - \sqrt{1 - \frac{1}{1 + x^2}}\right) + 2\,ab \sqrt{\left(1 \pm \sqrt{1 - \frac{1}{1 + x^2}}\right)\left(1 \mp \sqrt{1 - \frac{1}{1 + x^2}}\right)} \cos \theta \right\}$$

Simplification of the last term reduces to

$$R^2_{max} = \frac{1}{2 \sin^2 \theta}\left\{ a^2 \left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right) + b^2 \left(1 - \sqrt{1 - \frac{1}{1 + x^2}}\right) + \frac{2\,ab \cos \theta}{\sqrt{1 + x^2}} \right\} \qquad \text{(H-12)}$$

Similarly, for a > b

$$R^2_{min} = \frac{1}{2\sin^2\theta}\left\{a^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)-\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\} \tag{H-13}$$

and for b > a

$$R^2_{max} = \frac{1}{2\sin^2\theta}\left\{a^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)+\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\} \tag{H-14}$$

$$R^2_{min} = \frac{1}{2\sin^2\theta}\left\{a^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)-\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\} \tag{H-15}$$

If the angle of shift of the pseudo-major axis is ψ which is measured from that axis, m or n, to the true major x-axis, this angle can be calculated from the above information. Basically we are dealing with: for a > b.

$$\cos\psi = \frac{\text{The n value of } R_{max}}{R_{max}} \tag{H-16}$$

$$\cos^2\psi = \frac{a^2/2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)}{\frac{1}{2\sin\theta}\left\{a^2\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)+b^2\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+\frac{2ab\cos\theta}{\sqrt{1+x^2}}\right\}}$$

which reduces to

$$\cos^2\psi = \frac{\sin^2\theta\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)}{1+\sqrt{1-\frac{1}{1+x^2}}+b^2/a\left(1-\sqrt{1-\frac{1}{1+x^2}}\right)+\frac{2\,b/a\cos\theta}{\sqrt{1+x^2}}} \tag{H-17}$$

Similarly for $b > a$

$$\cos^2 \psi = \frac{\sin^2 \theta \left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right)}{a^2/b^2 \left(1 \sqrt{1 - \frac{1}{1 + x^2}}\right) + \left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right) + \frac{2a/b \cos \theta}{\sqrt{1 + x^2}}} \qquad \text{(H-16)}$$

Briefly, regarding symmetry, if we had started the original solution in terms of

$$\frac{dR}{dm} = 0$$

the resulting solution would have given

$$\frac{m^2}{b^2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\, a^2/b^2 \cos^2 \theta}{4 \frac{a^2}{b^2} \cos^2 \theta_x + \left(1 - a^2/b^2\right)}}\right)$$

Then let

$$y = \frac{1 - a^2/b^2}{2\, a/b \cos \theta} = \frac{b/a - a/b}{2 \cos \theta}$$

which would become

$$\frac{m^2}{b^2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{1 + y^2}}\right)$$

From this

$$\frac{n^2}{a^2} = 1 - \frac{m^2}{b^2} = \frac{1}{2}\left(1 \mp \sqrt{1 - \frac{1}{1 + y^2}}\right)$$

but we also have the $dR/dn = 0$ derivation that

$$\frac{n^2}{a^2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

Therefore

$$\frac{1}{2}\left(1 \mp \sqrt{1 - \frac{1}{1 + y^2}}\right) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{1 + x^2}}\right)$$

Continuing this derivation gives

$$x = \pm y$$

which for $x = -y$ substantiates the condition obtained in the derivations

$$x = \frac{1}{2 \cos \theta_x} (a/b - b/a)$$

$$y = \frac{1}{2 \cos \theta_x} \left(\frac{b}{a} - \frac{a}{b}\right) = -\frac{1}{2 \cos \theta_x} \left(\frac{a}{b} - b/a\right)$$

It is desirable to provide transforms for the major and minor axes such that, given values a, b, and θ_x, the true values of the major and minor axes and the orientation of these axes can be determined directly and simply. Such a transform would be the ratio of values of the true major axis to the pseudomajor axis a or b. Since the original maximum axis value could be a or b, we will call this R_{max}, and the true value, R'_{max}. The transform (squared) then becomes

for $a > b$

$$\left(\frac{R'_{max}}{R_{max}}\right)^2 = \frac{(R'_{max})^2}{a^2} = \frac{1}{2 \sin^2 \theta} \left\{ 1 + \sqrt{1 - \frac{1}{1 + x^2}} + \frac{b^2}{a^2}\left(1 - \sqrt{1 - \frac{1}{1 + x^2}}\right) + \frac{2b/a \cos \theta_x}{\sqrt{1 + x^2}} \right\} \quad \text{(H-19)}$$

and for $b > a$

$$\left(\frac{R'_{max}}{R_{max}}\right)^2 = \frac{(R'_{max})^2}{b^2} = \frac{1}{2 \sin^2 \theta} \left\{ \frac{a^2}{b^2}\left(1 - \sqrt{1 - \frac{1}{1 + x^2}}\right) + \left(1 + \sqrt{1 - \frac{1}{1 + x^2}}\right) + \frac{2a/b \cos \theta}{\sqrt{1 + x^2}} \right\} \quad \text{(H-20)}$$

It can be seen by inspection of these formulas that if the b/a of Equation (H-19) equals the a/b of Equation (H-20), then the x of Equation (H-20) will be the negative of the x of Equation (H-19), but the squares of these two will be equal

$$(-x_{H-20})^2 = (+x_{H-19})^2$$

Hence the transform functions are equal for given values of a/b, b/a, and θ_x. It is most convenient in visualizing these transformations to picture the pseudoellipse as having its major axis coincident with the corresponding axis in the oblique system m, n. Thus if a > b, we would draw the ellipse to be transformed with n as the major axis. Likewise if b > a, m would be considered the major axis of the pseudoellipse. By similar development the square of the minor axis transforms becomes

$$\left(\frac{R'_{min}}{R_{min}}\right)^2 = \frac{(R'_{min})^2}{b^2} = \frac{1}{2\sin^2\theta}\left\{\frac{a^2}{b^2}\left(1-\sqrt{1-\frac{1}{1+x^2}}\right) + \left(1+\sqrt{1-\frac{1}{1+x^2}}\right) - \frac{2a/b\cos\theta}{\sqrt{1+x^2}}\right\} \tag{H-21}$$

and for b > a

$$\left(\frac{R'_{min}}{R_{min}}\right)^2 = \frac{(R'_{min})^2}{a^2} = \frac{1}{2\sin^2\theta}\left\{1+\sqrt{1-\frac{1}{1+x^2}} + \frac{b^2}{a^2}\left(1-\sqrt{1-\frac{1}{1+x^2}}\right) - \frac{2b/a\cos\theta}{\sqrt{1+x^2}}\right\} \tag{H-22}$$

and since the same conditions prevail with respect to the x^2 values, the symmetrical equality of these two equations is also established.

Returning to the determination of ψ, it can be seen by comparing Equations (H-17) and (H-19) that

$$\cos^2\psi = \frac{1+\sqrt{1-\frac{1}{1+x^2}}}{2(\text{Major Axis XFrm})^2} \tag{H-23}$$

And finally,

$$\cos\psi = \frac{\sqrt{\frac{1}{2}\left(1+\sqrt{1-\frac{1}{1+x^2}}\right)}}{XF_{max}} \tag{H-24}$$

APPENDIX I

DERIVATION OF THE TRUE ELLIPSE TRANSFORMS FOR THREE OBSERVERS

As in the case of four observers, this derivation starts with the value of R as given by equation (F-26)

$$R^2 = \frac{1}{\sin^2 \theta_x} (m^2 + n^2 - 2mn \cos \theta_x) \qquad \text{(I-1)}$$

It is shown in Appendix J that there exists an elliptical, mathematical relationship of a constant contour of probability

$$\frac{n^2}{a^2} + \frac{m^2}{b^2} - \frac{mn}{ab} = 1 \qquad \text{(I-2)}$$

that is

$$n^2 = \frac{a}{2b} (m \pm \sqrt{4b^2 - 3m^2} \qquad \text{(I-3)}$$

giving, by substitution,

$$R^2 = \frac{1}{\sin^2 \theta} \left[m^2 + \frac{a^2}{4b^2} \left(m \pm \sqrt{4b^2 - 3m^2} \right)^2 - m \frac{a}{b} \left(m \pm \sqrt{4b^2 - 3m^2} \right) C \right] \qquad \text{(I-4)}$$

where

$$C = \cos \theta_x$$

$$R = \frac{1}{\sin \theta} \sqrt{m^2 + \frac{a^2}{4b^2} \left(m \pm \sqrt{4b^2 - 3m^2} \right)^2 - m \frac{a}{b} \left(m \pm \sqrt{4b^2 - 3m^2} \right) C} \qquad \text{(I-5)}$$

Again proceeding as in the case of independent observers, R_{max} and R_{min} are obtained when dR = 0. Since dn or dm is independent of dR, we are also solving for

$$\frac{dR}{dm} = 0$$

$$\frac{dR}{dm} = \frac{1}{2} (R \sin \theta)^{-1/2} \left[2m + \frac{a^2}{4b^2} \times 2(m + \sqrt{4b^2 - 3m^2}) \left(1 - \frac{6m}{2\sqrt{4b^2 - 3m^2}} \right) \right.$$

$$\left. - m \frac{a}{b} \left(1 + \frac{-6m}{2\sqrt{4b^2 - 3m^2}} \right) C - \frac{a}{b} \left(m + \sqrt{4b^2 - 3m^2} \right) C \right] = 0 \qquad \text{(I-6)}$$

Solving for m proceeds as follows.

To reduce repetition, let $x = \sqrt{4b^2 - 3m^2}$

Then, from equation (I-6),

$$2m + \frac{a^2}{2b^2}(m + x)\left(1 - \frac{3m}{x}\right) - m\frac{a}{b}\left(1 - \frac{3m}{x}\right)C - \frac{a}{b}(m + x)C = 0 \qquad \text{(I-7)}$$

$$2m + \left(1 - \frac{3m}{x}\right)\left[\frac{a^2}{2b^2}(m + x) - m\frac{a}{b}C\right] - \frac{a}{b}(m + x)C = 0$$

$$2mx + (x - 3m)\left[\frac{a^2}{2b^2}(m + x) - m\frac{a}{b}C\right] - \frac{a}{b}C(mx + x^2) = 0$$

$$2mx + \frac{a^2}{2b^2}(mx + x^2) - \frac{a}{b}C\,mx - \frac{3a^2}{2b^2}(m^2 + mx) + \frac{3ac}{b}m^2 - \frac{a}{b}C(mx + x^2) = 0$$

$$2mx + \frac{a^2}{2b^2}mx + \frac{a^2}{2b^2}x^2 - \frac{a}{b}C\,mx - \frac{3}{2}\frac{a^2}{b^2}m^2 - \frac{3}{2}\frac{a^2}{b^2}mx + 3\frac{a}{b}Cm^2$$
$$- \frac{a}{b}C\,mx - \frac{a}{b}C\,x^2 = 0$$

$$x(2m + \frac{a^2}{b^2}m - \frac{a}{b}C\,m - \frac{3}{2}\frac{a^2}{b^2}m - \frac{a}{b}C\,m) + x^2\left(\frac{a^2}{2b^2} - \frac{a}{b}C\right) +$$
$$m^2\left(3\frac{a}{b}C - \frac{3}{2}\frac{a^2}{b^2}\right) = 0$$

$$xm\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right) + x^2\left(\frac{a^2}{2b^2} - \frac{a}{b}C\right) + m^2\left(3\frac{a}{b}C - \frac{3}{2}\frac{a^2}{b^2}\right) = 0$$

$$xm\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right) + x^2\left(\frac{a^2}{2b^2} - \frac{a}{b}C\right) - 3m^2\left(\frac{a^2}{2b^2}C - \frac{a}{b}C\right) = 0$$

$$xm\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right) + (x^2 - 3m^2)\left(\frac{a^2}{2b^2} - \frac{a}{b}C\right) = 0$$

Replacing x^2 gives

$$xm\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right) + (4b^2 - 6m^2)\left(\frac{a^2}{2b^2} - \frac{a}{b}C\right) = 0$$

$$xm\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right) - (3m^2 - 2b^2)\left(\frac{a^2}{b^2} - 2\frac{a}{b}C\right) = 0 \tag{I-8}$$

Squaring gives

$$x^2m^2\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right)^2 = (3m^2 - 2b^2)^2\left(\frac{a^2}{b^2} - 2\frac{a}{b}C\right)^2 \tag{I-9}$$

Again replacing x^2

$$(4b^2 - 3m^2)\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right)^2 m^2 = (2b^2 - 3m^2)^2\left(\frac{a^2}{b^2} - 2\frac{a}{b}C\right)^2 \tag{I-10}$$

$$\frac{(2b^2 - 3m^2)^2}{(4b^2 - 3m^2)m^2} = \left(\frac{2 - \frac{2a}{b}C - \frac{a^2}{b^2}}{\frac{a^2}{b^2} - 2\frac{a}{b}C}\right)^2 \tag{I-11}$$

Let

$$k^2 = \left(\frac{2 - 2\frac{a}{b}C - \frac{a^2}{b^2}}{\frac{a^2}{b^2} - 2\frac{a}{b}C}\right)^2 \tag{I-12}$$

$$v = m^2$$

Then

$$(2b^2 - 3v)^2 = k^2v(4b^2 - 3v) \tag{I-13}$$

$$4b^4 - 12b^2v + 9v^2 = 4k^2b^2v - 3k^2v^2$$

$$3(3 + k^2)v^2 - 4b^2(3 + k^2)v + 4b^4 = 0$$

$$3v^2 - 4b^2v + \frac{4b^4}{3 + k^2} = 0$$

$$v = m^2 = \frac{4b^2 \pm \sqrt{16b^4 - 48\frac{b^4}{3 + k^2}}}{6}$$

$$m^2 = \frac{2}{3}b^2\left(1 \pm \sqrt{1 - \frac{1}{1 + k\frac{2}{3}}}\right) \tag{I-14}$$

Finally to get this into a form comparable to the four-observer case, let

$$\xi^2 = \frac{k^2}{3} = \frac{\left(2 - 2\frac{a}{b}C - \frac{a^2}{b^2}\right)^2}{3\left(\frac{a^2}{b^2} - 2\frac{a}{b}C\right)^2} \tag{I-15}$$

giving

$$m^2 = \frac{2}{3}b^2\left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right) \tag{I-16}$$

as the coordinates of points on the true ellipse which are the end points of the major and minor axes.

The n coordinates are obtained directly from considerations of symmetry. If initial substitution in equation (I-6) had been made for m instead of n, and the equation

$$\frac{dR}{dn} = 0$$

solved for n, the resulting equations are the same as those above if, also, the following replacements are made

$$m \rightarrow n$$
$$n \rightarrow m$$
$$a \rightarrow b$$
$$b \rightarrow a$$

Then

$$n^2 = \frac{2}{3} a^2 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right) \qquad \text{(I-17)}$$

where

$$\xi^2 = \frac{\left(2 - 2\frac{b}{a} C - \frac{b^2}{a^2}\right)^2}{3\left(\frac{b^2}{a^2} - 2\frac{b}{a} C\right)^2} \qquad \text{(I-18)}$$

For a quick demonstration of the correctness of these equations, consider the special case of

$$\theta = 90^\circ \text{ (i.e., orthogonal axes)}$$

$$a = b$$

$$\xi^2 = \frac{(2 - 0 - 1)^2}{3(1 - 0)^2} = \frac{1}{3}$$

$$m^2 = \frac{2}{3} b^2 (1 \pm \tfrac{1}{2})$$

$$m = \pm \frac{b}{\sqrt{3}} \text{ and } \pm b$$

and

$$n = \pm \frac{a}{\sqrt{3}} \text{ and } \pm a$$

Remembering that when $a = b$,

$$\frac{m^2}{b^2} + \frac{n^2}{a^2} - \frac{mn}{ab} = 1,$$

is a skewed ellipse of 45° given by

$$\frac{(m')^2}{(b')^2} + \frac{(n')^2}{(a')^2} = 1 \qquad \text{(I-19)}$$

where

$$(b')^2 = \frac{1}{\frac{\cos^2 45^\circ}{b^2} + \frac{\cos^2 45^\circ}{b^2} - \frac{\sin 45^\circ \cos 45^\circ}{b^2}} \qquad \text{(I-20)}$$

$$= \frac{1}{\frac{1}{2b^2} + \frac{1}{2b^2} - \frac{1}{2b^2}} = 2b^2$$

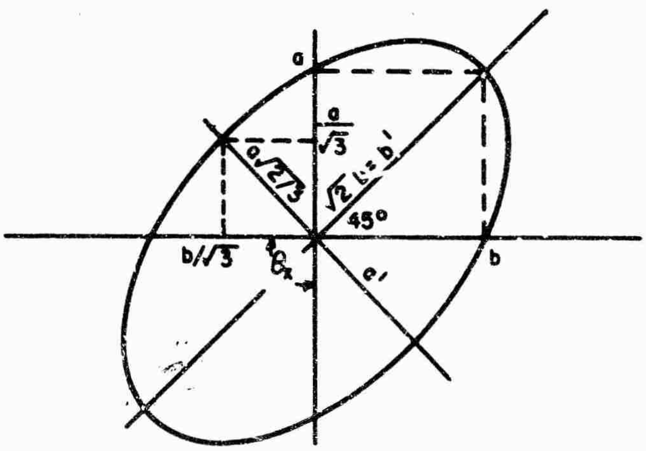
$$b' = \sqrt{2}\, b \qquad \text{(I-21)}$$

and

$$(a')^2 = \frac{1}{\frac{1}{2a^2} + \frac{1}{2a^2} + \frac{1}{2a^2}} = \frac{2}{3} b^2 \qquad \text{(I-22)}$$

$$a' = \sqrt{\frac{2}{3}}\, a \qquad \text{(I-23)}$$

These values are shown on the following sketch which illustrates and verifies the special case.



Another, more typical, example was worked out analytically and then checked with graphical solution. The parameters selected were

$$\sigma_n = 0.25$$

$$\sigma_m = 0.6$$

$$s = 1$$

giving

$$a = s\sqrt{\frac{3}{2}}\,\sigma_n = 0.306$$

$$b = s\sqrt{\frac{3}{2}}\,\sigma_m = 0.735$$

We then wish to find the locus of the points

$$\frac{m^2}{(0.735)^2} + \frac{n^2}{(0.306)^2} - \frac{mn}{(0.735 \times 0.306)} = 1$$

onto the oblique axes m, n, θ_x. The graphical solution is shown in Figure I-1. The plotted values were obtained from the relationship

$$n = \frac{a}{2b}\left(m \pm \sqrt{4b^2 - 3m^2}\right)$$

The coordinates from this graph for the terminus of R_{min} and R_{max} are

for R_{max}: $m = 0.780$, $n = 0.043$

for R_{min}: $m = 0.325$, $n = 0.348$

To solve for the coordinates analytically

$$\frac{a}{b} = 0.416, \qquad \cos 20^\circ = 0.94$$

$$\xi = \frac{2 - 2 \times 0.416 \times 0.94 - 0.173}{\sqrt{3}\,(0.173 - 2 \times 0.416 \times 0.94)} = -\frac{1.71}{\sqrt{3}}$$

$$\xi^2 = 0.975$$

$$m^2 = \frac{2}{3} b^2 \left(1 \pm \sqrt{1 - \frac{1}{1 + 0.975}}\right) = \frac{2}{3}\,\overline{0.705}^2 (1 \pm 0.703)$$

$$m = 1.063b,\ 0.445b$$

$$m = 0.782,\ 0.327$$

Likewise, for n

$$\zeta = \frac{2 - 2 \times 2.4 \times 0.94 - 5.76}{\sqrt{3}\,(5.76 - 2 \times 2.4 \times 0.94)} = -3.82$$

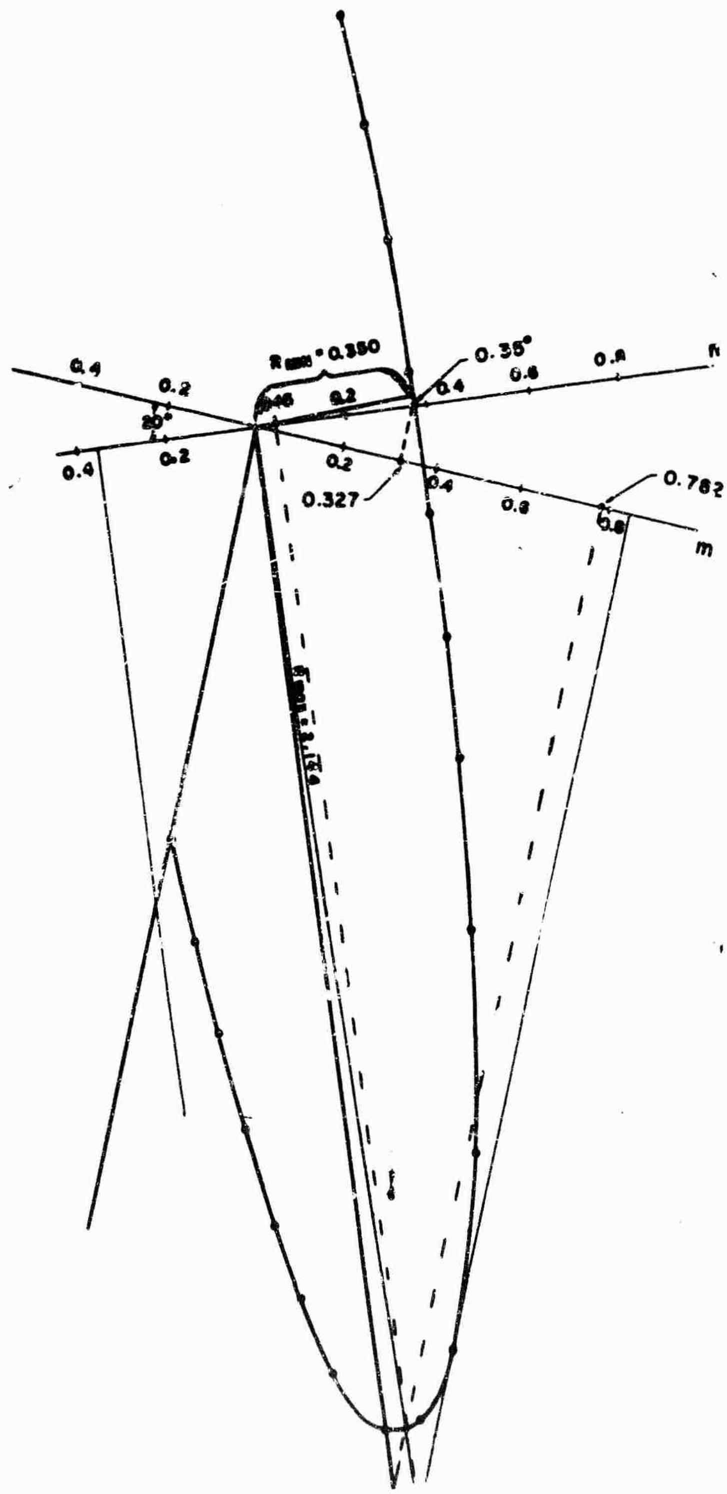


Figure I-1. An Example of an EEP Problem on the Oblique Axis

$$\xi^2 = 14.6$$

$$n^2 = \frac{2}{3} a^2 \left(1 \mp \sqrt{1 - \frac{1}{15.6}}\right) = \frac{2}{3} \times \overline{0.306}^2 (1 \mp 0.968)$$

$$n = 0.146a,\ 1.145a$$

$$n = 0.0447,\ 0.350$$

And the coordinates become

R_{max}: $m = 0.782$, $n = 0.0447$

R_{min}: $m = 0.327$, $n = 0.350$

Therefore, we are dealing with something less than 2% error, including graphical and analytical calculations, and the formulas are verified for this example.

Providing transforms for the true values of the major and minor axes is more complicated than for the previous case. First, the quantities a, b no longer represent the major and minor axes of the pseudo-orthogonal ellipse. They still, however, represent convenient quantities for determining the desired transforms.

Returning to the general form of the radial vector to the true ellipse, and substituting the derived values for the true coordinates m, n, gives

$$R^2 = \frac{1}{\sin^2 \theta_x} \left\{ \frac{2}{3} b^2 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right) + \frac{2}{3} a^2 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right) \pm \right.$$

$$\left. \pm \frac{4}{3} ab \sqrt{\left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right) \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right)} \cos \theta_x \right\} \qquad \text{(I-24)}$$

Defining the transforms

$$XF_{Max} = \frac{R}{a}\ (a > b) = \frac{R}{b}\ (b > a) \qquad \text{(I-25)}$$

$$XF_{Min} = \frac{R}{b}\ (a > b) = \frac{R}{a}\ (b > a) \qquad \text{(I-26)}$$

The needed relationships R/a and R/b are determined from

$$\frac{R^2}{a^2} = \frac{1}{\sin^2\theta}\left\{\frac{2}{3}\frac{b^2}{a^2}\left(1 \pm \sqrt{1 - \frac{1}{1+\xi^2}}\right) + \frac{2}{3}\left(1 \pm \sqrt{1 - \frac{1}{1+\zeta^2}}\right) \pm \right.$$

$$\left. \pm \frac{4}{3}\frac{b}{a}\sqrt{\left(1 \pm \sqrt{1 - \frac{1}{1+\xi^2}}\right)\left(1 \pm \sqrt{1 - \frac{1}{1+\zeta^2}}\right)}\, C\right\} \qquad \text{(I-27)}$$

$$\frac{R^2}{b^2} = \frac{1}{\sin^2\theta}\left\{\frac{2}{3}\left(1 \pm \sqrt{1 - \frac{1}{1+\xi^2}}\right) + \frac{2}{3}\frac{a^2}{b^2}\left(1 \pm \sqrt{1 - \frac{1}{1+\zeta^2}}\right) \pm \right.$$

$$\left. \pm \frac{4}{3}\frac{a}{b}\sqrt{\left(1 \pm \sqrt{1 - \frac{1}{1+\xi^2}}\right)\left(1 \pm \sqrt{1 - \frac{1}{1+\zeta^2}}\right)}\, C\right\} \qquad \text{(I-28)}$$

Since the key factors to these solutions are the quantities

$$\sqrt{1 - \frac{1}{1+\xi^2}} \quad \text{and} \quad \sqrt{1 - \frac{1}{1+\zeta^2}},$$

and since one abscissa can provide either a/b of ξ or b/a of ζ, Figure I-2 is a representation of either radical. The selection of the parameter θ_x gives samples through the crossing angle spectrum.

The maxima (of unity value) are obtained for

$$b/a \text{ or } a/b = 2C \; (2 \cos \theta_x)$$

The minima (of zero value) are obtained for

$$b/a \text{ or } a/b = -C + \sqrt{C^2 + 2}$$

The case for $\theta = 60^0$ is very interesting and perhaps even anomalous. The full significance of this curve has not been explored for this document except that it passed all validity or application tests developed during this writing.

The next task is that of making the various polarity decisions associated with the radicals. There are two distinct decisions to be made: (1) the polarity of the radical in determining the value of the coordinates, and (2) the relative value of the coordinates themselves with respect to each other. This is a much more difficult task than for the independent case of only one radical value per a, b, c.

The tools and criteria for making these decisions were

a. Figure I-3

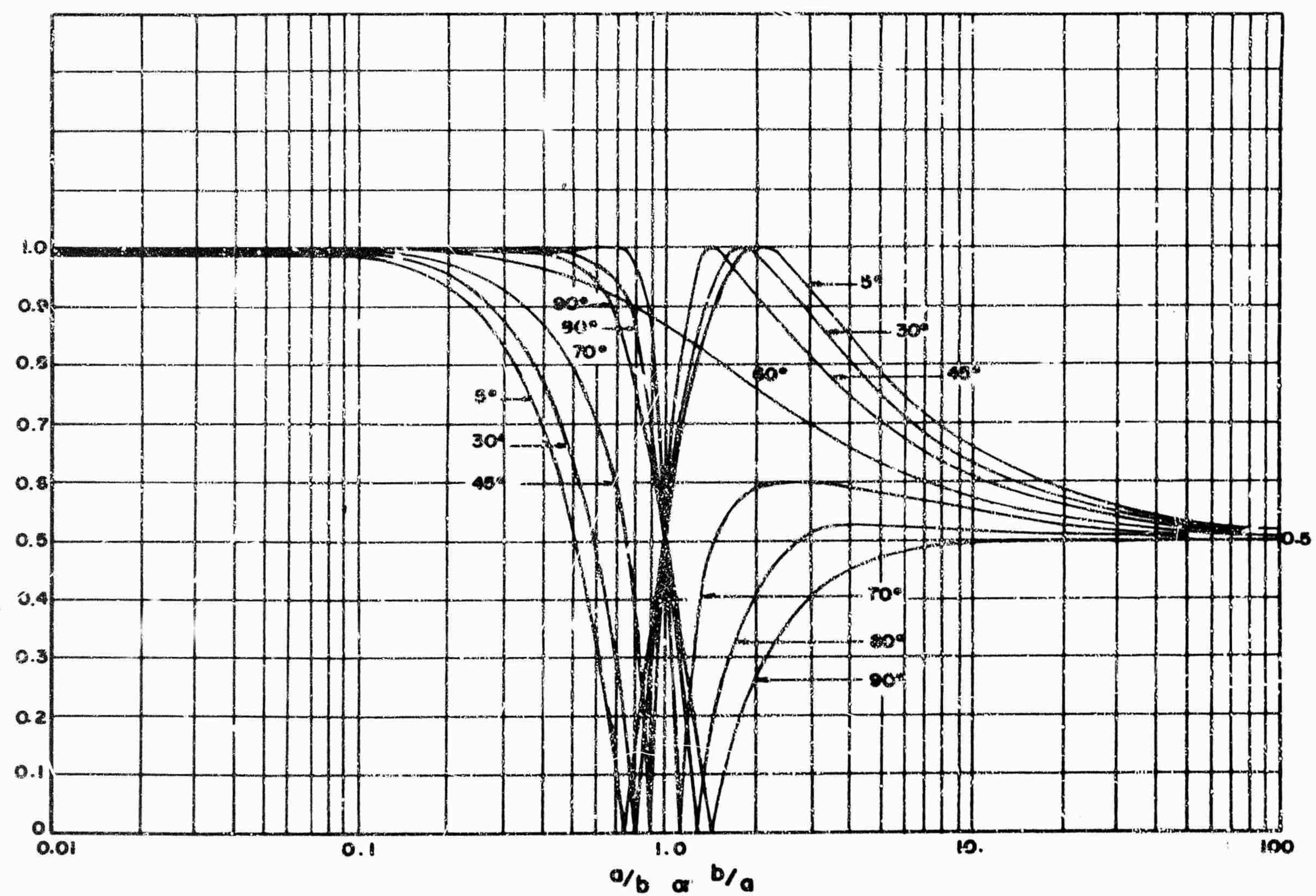


Figure I-2. Curves of $\sqrt{1 - \frac{1}{1 + \xi^2}}$ vs. a/b

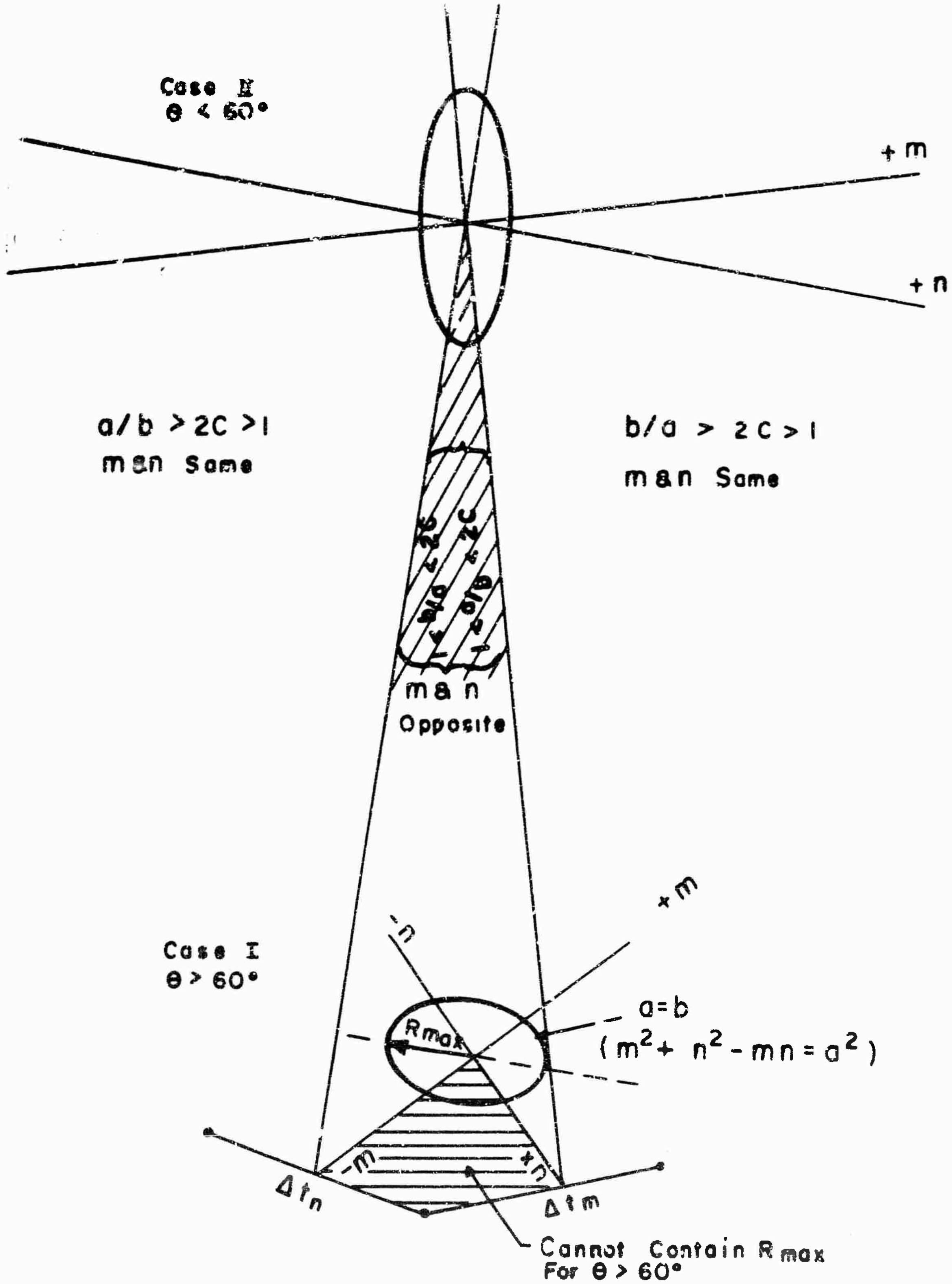


Figure I-3. Geometrical Polarity Decision Criteria

b. $\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{mn}{ab} = 1$

c. Some trial and error

The results are tabulated in Table I-1 which seemed to be about the best, though by no means the only, way of depicting these polarity decisions. Since the quantities

$$2C \text{ and } -C + \sqrt{C^2 + 2}$$

can both be greater or less than one, all possible cases are charted. The idea used in deriving the chart was to find a starting point for which all the factors were known and then reason what must happen from then on. (The author is sure there is a purer mathematical or classical way of doing this, but the idea was successful.) The starting point used is b = a. Examination of the elliptical equation shows that the choice of radical polarities is limited to the extent that they must be the same for m and n at b = a. The next significant factor, obtained by examining Table I-1, is that $\theta_x = 60^o$ is a dividing point for these polarities and the cases of $\theta_x > 60^o$ must be examined separately from $\theta_x < 60^o$. Finally, the separation of m and n for XFMax and XFMin is obtained by almost any test points within the regions specified.

Having thus obtained a complete initial set of polarities for the radicals at b = a, the rest is the intelligent guess that as

$$\sqrt{1 - \frac{1}{1 + (\xi^2 \text{ or } \zeta^2)}}$$

goes through zero at

$$-C + \sqrt{C^2 + 2},$$

the polarity associated with the radical must reverse as a/b and b/a are varied through their interlocked values.

The significant factor in determining the polarities of m and n relative to each other is the value 2C, for at a/b or b/a = 2C, m or n are, respectively, zero. This would be intuitively the point at which a coordinate's polarity would change. Verification of this change, and determination of the manner in which the polarities changed, were accomplished by graphical analysis, with the results shown in Table I-2. It should be remembered that as far as the transform formulas are concerned, we need know only whether the relative polarities are the same or opposite.

Determination of the angle of major axis shift ψ is performed in a manner similar to the analysis for four observers. It proceeds as follows

For b > a

$$\cos^2 \psi = \frac{[\text{value of m(for } R_{Max})]^2}{R^2_{Max}}$$

$$= \frac{2/3\, b^2 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right)}{b^2 \times F^2_{Max}}$$

$$= \frac{2/3 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right)}{XF^2_{Max}}$$

or

$$\cos \psi = \frac{\sqrt{2/3 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right)}}{XF_{Max}}$$

where the polarity of the radical is determined in the same manner (using Table I-1) as for the XF_{Max}. The angle ψ would then be measured from the m axis.

Similarly, for a > b

$$\cos \psi = \frac{\sqrt{2/3 \left(1 \pm \sqrt{1 - \frac{1}{1 + \xi^2}}\right)}}{XF_{Max}}$$

and ψ would be measured from the n axis.

TABLE I-1

POLARITY DECISION CRITERIA FOR THE RADICAL

DETERMINATION OF RADICAL POLARITY

$$\left(x = -C + \sqrt{C^2 + 2}\right)$$

			0 – x	x – 1	1 – ∞
$\theta < 60°$	XF_{Max}	m	← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ → a/b	$\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	b/a
		n			← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →
		m	b/a		← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ → a/b
		n	← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →	$\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	
	XF_{Min}	m	← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ → a/b	$\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	b/a
		n			← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →
		m	b/a		← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ → a/b
		n	← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →	$\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	

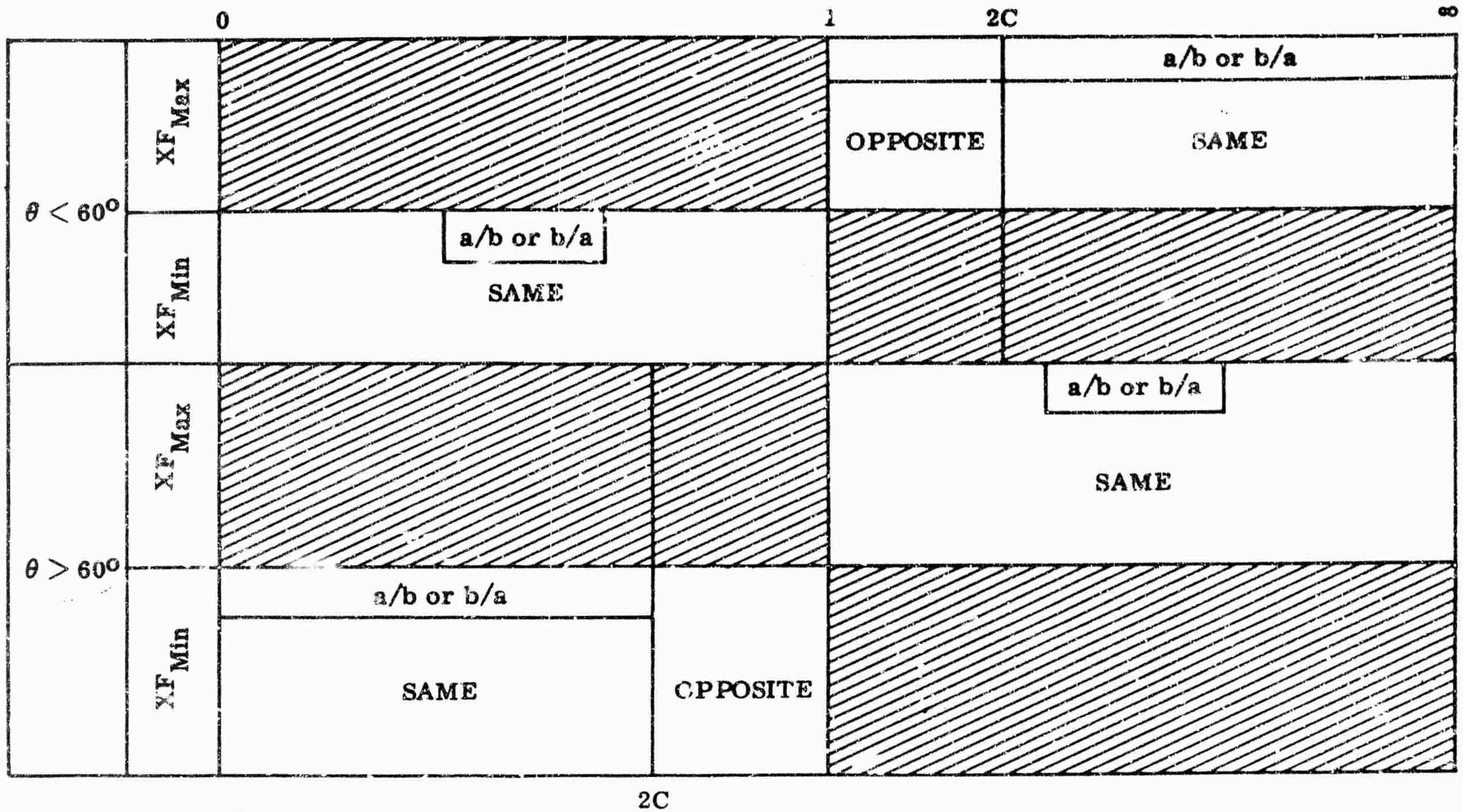
			0 – 1	1 – x	x – ∞
$\theta > 60°$	XF_{Max}	m	← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ → a/b		b/a
		n		$\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →
		m	b/a	$\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	a/b ← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →
		n	← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →		
	XF_{Min}	m	← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ → a/b		b/a
		n		$\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →
		m	b/a	$\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$	a/b ← $\left(1 + \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →
		n	← $\left(1 - \sqrt{1 - \frac{1}{1+\xi^2}}\right)$ →		

TABLE I-2

POLARITY DECISION CRITERIA FOR THE COORDINATES

DETERMINATION OF RELATIVE COORDINATE POLARITY

(Shaded Areas are Redundant)



APPENDIX J

DERIVATION OF THE JOINT PROBABILITY DENSITY FUNCTION FOR THREE OBSERVERS

The basic problem is to find the true degree of dependency or correlation between the time difference pairs which have one observer in common. Suppose observer B were the center or common observer. Further, for notation, let the errors or deviations from true time readings be

$$T_A - t_A = \Delta t_A \qquad \text{(J-1)}$$

$$T_B - t_B = \Delta t_B \qquad \text{(J-2)}$$

$$T_C - t_C = \Delta t_C \qquad \text{(J-3)}$$

where T_A, T_B and T_C are the true instantaneous values of time of arrival and t_A, t_B, t_C are the observed instantaneous values.

Figure J-1 presents one example of a possible time configuration with probability density functions sketched for each observer. The probability that A has made an error of magnitude between Δt_A and $\Delta T_A + d\Delta t_A$ is

$$p(\Delta t_A) = \frac{1}{\sigma_A \sqrt{2\pi}} \exp\left[-\frac{\Delta T_A^{\ 2}}{2\sigma_A^{\ 2}}\right] d\Delta t_A \qquad \text{(J-4)}$$

Similarly,

$$p(\Delta t_B) = \frac{1}{\sigma_B \sqrt{2\pi}} \exp\left[-\frac{\Delta T_B^{\ 2}}{2\sigma_B^{\ 2}}\right] d\Delta t_B \qquad \text{(J-5)}$$

$$p(\Delta t_C) = \frac{1}{\sigma_C \sqrt{2\pi}} \exp\left[-\frac{\Delta T_C^{\ 2}}{2\sigma_C^{\ 2}}\right] d\Delta t_C \qquad \text{(J-6)}$$

Defining the quantities Δt_N and Δt_M as

$$\Delta t_N = (T_A - T_B) - (t_A - t_B) \qquad \text{(J-7)}$$

$$\Delta t_M = (T_B - T_C) - (t_B - t_C) \qquad \text{(J-8)}$$

the goal of this derivation is to determine the joint probability function $f(\Delta t_N, \Delta t_M)$.

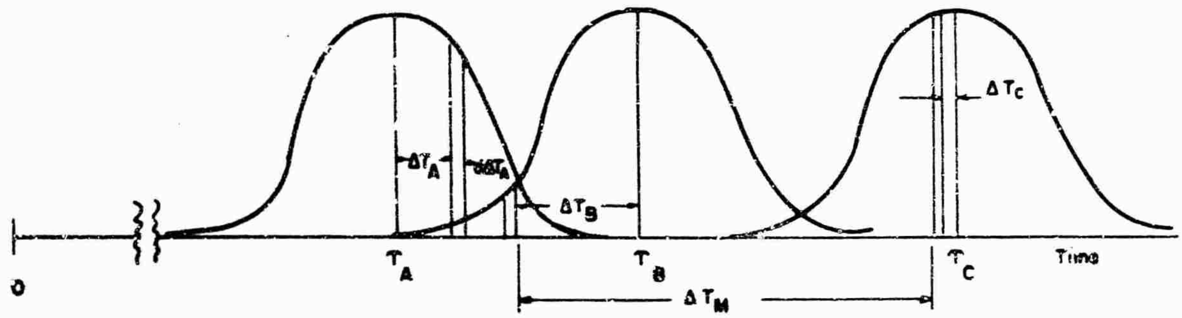


Figure J-1. An Example of Probability Distributions for Three Observers

Substituting from equations (J-1) and (J-2)

$$\Delta t_N = \Delta t_A - \Delta t_B \tag{J-9}$$

$$\Delta t_M = \Delta t_B - \Delta t_C \tag{J-10}$$

and, finally,

$$\Delta t_A = \Delta t_N + \Delta t_B \tag{J-11}$$

$$\Delta t_C = \Delta t_B - \Delta t_M \tag{J-12}$$

Considerable confusion and discussion can arise on the relative polarities of these quantities and, as indicated in Figure J-1, the possibilities include all combinations of polarities of the three quantities. Since we are concerned only with $\Delta t_A{}^2$ and $\Delta t_C{}^2$, these combinations can all be reduced to two: Δt_B is either aiding or opposing the quantities Δt_N, Δt_M.

The next important probabilistic concept is, that to obtain a specific Δt_N and Δt_M, we must also obtain specific values for Δt_A, Δt_B, and Δt_C. The probability of doing this is the probability of obtaining these three errors simultaneously, and

$$p(\Delta t_N, \Delta t_M) = p(\Delta t_A)\, p(\Delta t_B)\, p(\Delta t_C) \tag{J-13}$$

where Δt_A, Δt_B and Δt_C represent those quantities necessary to obtain a desired Δt_N and Δt_M.

Substituting from equations (J-4), (J-5), and (J-6)

$$p(\Delta t_N, \Delta t_M) = \frac{1}{\sigma_A \sigma_B \sigma_C (2\pi)^{3/2}} \exp\left[-\left(\frac{\Delta t_A{}^2}{2\sigma_A{}^2} + \frac{\Delta t_B{}^2}{2\sigma_B{}^2} + \frac{\Delta t_C{}^2}{2\sigma_C{}^2}\right)\right] d\Delta t_A\, d\Delta t_B\, d\Delta t_C \tag{J-14}$$

It is assumed at this point that characteristics of the three observers are so nearly alike that

$$\sigma_A = \sigma_B = \sigma_C = \sigma$$

and

$$p(\Delta t_N, \Delta t_M) = \frac{1}{\sigma^3 (2\pi)^{3/2}} \exp\left[-\frac{(\Delta t_A{}^2 + \Delta t_B{}^2 + \Delta t_C{}^2)}{2\sigma^2}\right] d\Delta t_A d\Delta t_B d\Delta t_C \qquad \text{(J-15)}$$

Now, substituting for Δt_A and Δt_C,

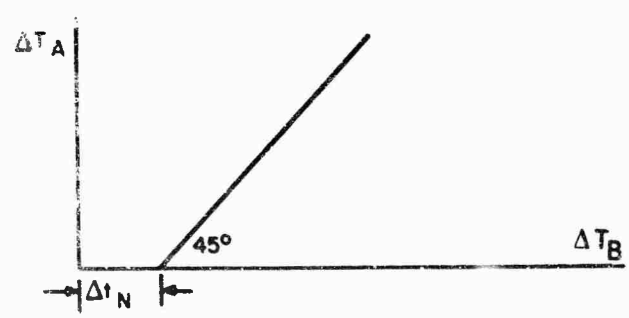
$$p(\Delta t_N, \Delta t_M) = \frac{1}{\sigma^3 (2\pi)^{3/2}} \exp -\left[\frac{\Delta t_B{}^2 + (\Delta t_N \pm \Delta t_B)^2 + (\Delta t_M \pm \Delta t_B)^2}{2\sigma^2}\right] d\Delta t_A d\Delta t_B d\Delta t_C \qquad \text{(J-16)}$$

To obtain the total probability for the occurrence of a given Δt_N and Δt_M, we must add the probabilities for all the mutually exclusive collectively exhaustive ways in which these Δt_N, Δt_M can be obtained. This is done by allowing Δt_B to successively take on all possible values from $-\infty$ to $+\infty$.

Thus

$$p(\Delta t_N, \Delta t_M) = \frac{d\Delta t_A d\Delta t_C}{\sigma^3 (2\pi)^{3/2}} \int_{-\infty}^{\infty} \exp -\left[\frac{\Delta t_B{}^2 + (\Delta t_N \pm \Delta t_B)^2 + (\Delta t_M \pm \Delta t_B)^2}{2\sigma^2}\right] d\Delta t_B \qquad \text{(J-17)}$$

Since Δt_A, Δt_B and Δt_C are independent and since, for example, Δt_A and Δt_B are kept separated by Δt_N,



$$\frac{d\Delta t_A}{d\Delta t_B} = 1$$

and we can conclude

$$d\Delta t_A = d\Delta t_B = d\Delta t_C = d\Delta t$$

Equation (J-17) can be rewritten as

$$\frac{p(\Delta t_N, \Delta t_M)}{(d\Delta t)^2} = \frac{1}{\sigma^3 (2\pi)^{3/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{\Delta t_B^{\,2} + (\Delta t_N \pm \Delta t_B)^2 + (\Delta t_M \pm \Delta t_B)^2}{2\sigma^2}\right] d\Delta t_B \qquad \text{(J-18)}$$

If we then define the joint probability density function as

$$f(\Delta t_N, \Delta t_M) \equiv \frac{p(\Delta t_N, \Delta t_M)}{(d\Delta t)^2} \qquad \text{(J-19)}$$

the result is

$$f(\Delta t_N, \Delta t_M) = \frac{1}{\sigma^3 (2\pi)^{3/2}} \int_{-\infty}^{\infty} \exp -\left[\frac{\Delta t_B^{\,2} + (\Delta t_N \pm \Delta t_B) + (\Delta t_M \pm \Delta t_B)^2}{2\sigma^2}\right] d\Delta t_B \qquad \text{(J-20)}$$

For those interested in a more classical verification of this relationship, the problem is one of

given $P(x, y, z)$

find $P'(u, v)$

where x, y, and z are independent variables and u and v are some functions as

$$u = u(x, y, z) \text{ and } v = v(x, y, z) \qquad \text{(J-21)}$$

For any particular u, v there is a curve in x, y, z space. The probability of being in the incremental area (u + du - u) by (v + dv - v) is also the probability of being in the volume x, y, z which is formed by the locus of u(x, y, z) and v(x, y, z) for each given u, v. If the curve is single-valued with respect to the z axis, for example, we can integrate over z to find the total probability of being in the volume. To find these curves as a function of z, the equation (J-21) must be solved to give

$$x = x(u, v, z) \qquad \text{(J-22)}$$

$$y = y(u, v, z) \qquad \text{(J-23)}$$

Then

$$P'dudv = \sum_{\text{All } z} P[x(u, v, z), y(u, v, z), z]dA \qquad (J-24)$$

where, using vector notation,

$$dA = \left|\left(\frac{\partial x}{\partial \mu}\vec{u}_x + \frac{\partial y}{\partial \mu}\vec{u}_y\right) \times \left(\frac{\partial x}{\partial v}\vec{v}_x + \frac{\partial y}{\partial v}\vec{v}_y\right)\right| dudv \qquad (J-25)$$

and $\vec{\mu}_x$, $\vec{\mu}_y$, $\vec{v}_x$, $\vec{v}_y$ are unit vectors. When the cross product is performed, this reduces to

$$dA = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \; dudv$$

Therefore

$$P' = \int P[x(u, v, z), y(u, v, z), z] \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dz \qquad (J-26)$$

where P' is the desired probability density function.

For the example at hand

$$P = \frac{1}{\sigma^3 (2\pi)^{3/2}} \exp\left[-\frac{x^2 + y^2 + z^2}{2\sigma^2}\right]$$

where

$$x = \Delta t_N + \Delta t_B$$

$$y = \Delta t_M + \Delta t_B$$

$$z = \Delta t_B$$

$$u = (x - z) = \Delta t_N + \Delta t_B - \Delta t_B = \Delta t_N$$

$$v = (y - z) = \Delta t_M + \Delta t_B - \Delta t_B = \Delta t_M$$

Evaluating

$$\frac{\partial x}{\partial u} = \frac{\partial(\Delta t_N + \Delta t_B)}{\partial \Delta t_N} = 1, \quad \frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = 0$$

gives

$$\left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| = 1$$

and

$$P' = \frac{1}{\sigma^3 (2\pi)^{3/2}} \int_{-\infty}^{\infty} \exp\left[- \frac{(\Delta t_N \pm \Delta t_B)^2 + (\Delta t_M \pm \Delta t_B)^2 + \Delta t_B^{\ 2}}{2\sigma^2} \right] d\Delta t_B \qquad \text{(J-27)}$$

which is precisely the same as equation (J-20).

Continuing with the solution of equation (J-20), when the exponent is expanded and collected, it can be written in the form

$$\frac{3}{2\sigma^2} \Delta t_B^{\ 2} + 2\left[\pm \frac{(\Delta t_N + \Delta t_B)}{2\sigma^2} \right] \Delta t_B + \frac{\Delta t_N^{\ 2} + \Delta t_M^{\ 2}}{2\sigma^2}$$

from the integral tables which state

$$\int_{-\infty}^{\infty} \exp\left[- (Ax^2 + Bx + C) \right] dx = \sqrt{\frac{\pi}{A}} \exp\left[\frac{B^2 - AC}{A} \right] \qquad \text{(J-28)}$$

Substituting

$$f(\Delta t_N, \Delta t_M) = \frac{1}{(2\pi)^{3/2}\sigma^3} \sqrt{\frac{\pi}{3/2\sigma^2}} \left\{ \exp\left[\frac{(\Delta t_N + \Delta t_M)^2}{4\sigma^4} - \frac{3}{2\sigma^2} \cdot \frac{\Delta t_N^{\ 2} + \Delta t_M^{\ 2}}{2\sigma^2} \right] \frac{1}{3/2\sigma^2} \right\}$$

(J-29)

when the coefficient and exponent are simplified

$$f(\Delta t_N, \Delta t_M) = \frac{1}{\pi 2\sigma^2 \sqrt{3}} \exp\left[- \frac{2/3(\Delta t_N^{\ 2} + \Delta t_M^{\ 2} - \Delta t_M \Delta t_N)}{2\sigma^2} \right] \qquad \text{(J-30)}$$

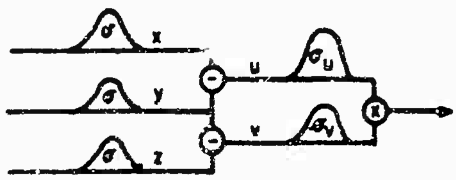
and using the standard deviation per pair of stations, as noted in Appendix E, e.g.,

$$\sigma_x^{\ 2} = 2\sigma^2 \qquad \text{(J-31)}$$

$$f(\Delta t_N, \Delta t_M) = \frac{1}{\sigma^2 \pi \sqrt{3}} \exp\left[- \frac{2/3(\Delta t_N^{\ 2} + \Delta t_M^{\ 2} - \Delta t_N \Delta t_M)}{\sigma_x^{\ 2}} \right] \qquad \text{(J-32)}$$

which is the formula discussed in the main text.

It is interesting to note the direct comparison between this result and the result of Gaussian noise adder circuits. If we have three Gaussian noise inputs, x, y, z with standard deviation σ for each, and use an adder (or subtracter) circuit element for x, y and z, the situation symbolically looks like



The density function for this condition, given in numerous texts, is

$$f(u,v) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\left[\frac{{}^2 - 2\rho uv + v^2}{(1-\rho^2)\sigma^2}\right]\right\} \tag{J-33}$$

where ρ is the correlation coefficient obtained from

$$\rho = \frac{\overline{uv}}{\sigma_u \sigma_v}$$

However

$$\overline{uv} = \overline{(y-x)(y-z)} = \overline{y^2} - \overline{xy} - \overline{yz} + \overline{xz}$$

$$\overline{uv} = \overline{y^2}$$

and

$$\overline{y^2} = \sigma^2 + \bar{y}^2 = \sigma^2$$

hence

$$\overline{uv} = \sigma^2$$

It can also be shown

$$\sigma_u = \sigma_v = \sqrt{2}\sigma$$

Hence

$$\rho = \frac{\sigma^2}{\sqrt{2}\sigma\sqrt{2}\sigma} = \frac{1}{2}$$

And, for this value of ρ,

$$f(u, v) = \frac{1}{\sqrt{3}\pi\sigma^2} \exp\left[- 2/3 \frac{(u^2 + v^2 - uv)}{\sigma^2} \right]$$

which, again, is identical to the derived formula.

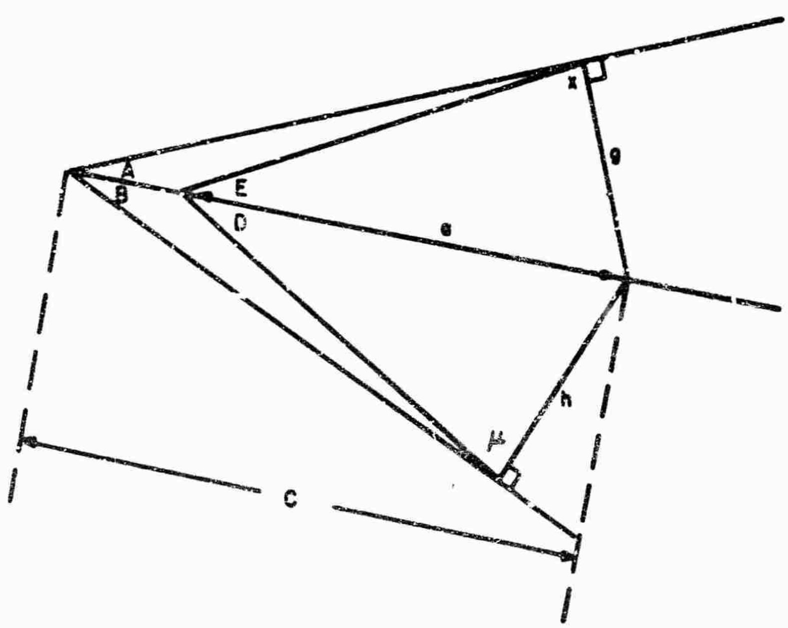
APPENDIX K

MODIFICATION OF ELLIPTICAL TRANSFORMS FOR A DIVERGING, OBLIQUE COORDINATE SYSTEM

As stated in the main text the true coordinate system about the point P is not a gridwork of lines parallel to the LOP's at the assumed true point P. Rather, each set of points m, n which satisfy the error conditions Δt_m, Δt_n, lie on the hyperbolas associated with Δt_m, Δt_n so that the angles between the m and n axes and the hyperbolas are right angles.

If we (see Figure K-1) also construct the intersections of $\pm m$, $\pm n$ in the usual manner on lines parallel to the LOP's, then these points and the points described above, for example m' and n', and the true point P all lie in a straight line. It is then desired to determine the change or adjustment of the values, called R (or C) in the text, into the values d and a. This is accomplished with engineering approximations in the following manner.

In addition to Figure K-1, consider the following enlargements of the regions θ_x and θ_y.



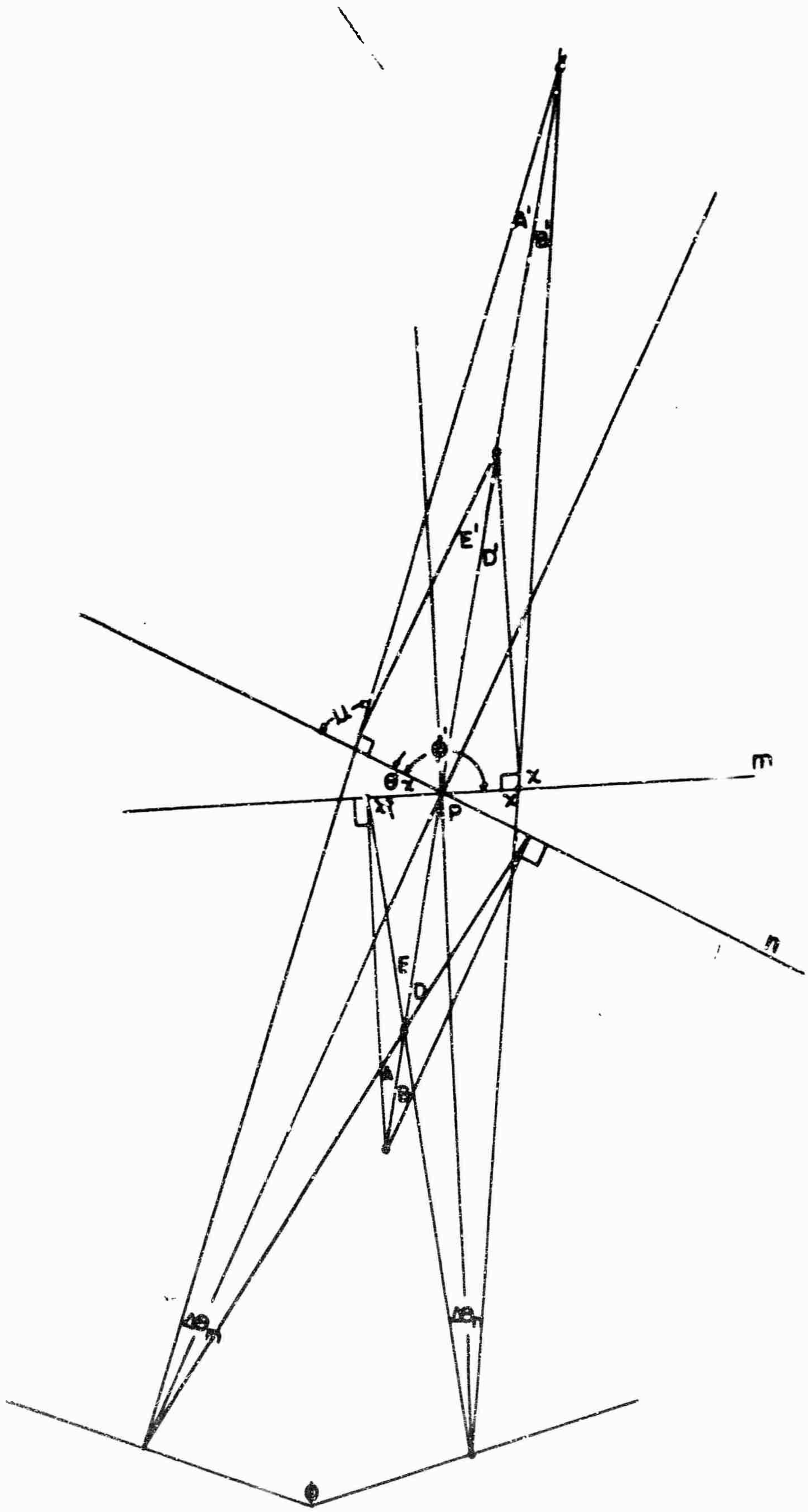


Figure K-1. Geometry of the True Diverging Oblique Axis System

$$\theta_x = A + B, \ \theta_y = E + D \tag{K-1}$$

Then by trigonometry

$$\sin A = \frac{g}{c}, \ \sin B = \frac{h}{c}$$

$$\frac{\sin A}{\sin B} = \frac{g}{h} \tag{K-2}$$

and

$$\frac{g}{\sin E} = \frac{a}{\sin x}, \ \frac{g}{\sin A} = \frac{c}{\sin 90}$$

$$\frac{h}{\sin D} = \frac{a}{\sin \mu}, \ \frac{h}{\sin B} = \frac{c}{\sin 90}$$

$$\frac{\sin E}{\sin D} = \frac{g}{h} \frac{\sin x}{\sin \mu} \tag{K-3}$$

We then exercise the engineering approximation that

$$\frac{\sin x}{\sin \mu} \simeq 1 \tag{K-4}$$

and

$$\frac{\sin E}{\sin D} = \frac{g}{h} \tag{K-5}$$

Further

$$\sin A \simeq A, \ \sin B \simeq B, \text{ etc.}$$

Therefore

$$\frac{A}{B} \simeq \frac{g}{h}, \ \frac{E}{D} \simeq \frac{g}{h} \tag{K-6}$$

$$\frac{A}{B} \simeq \frac{E}{D} \tag{K-7}$$

or

$$\frac{A + B}{B} \simeq \frac{E + D}{D} \tag{K-8}$$

$$\frac{B}{\theta_x} = \frac{D}{\theta_y} \tag{K-9}$$

Further, by the law of sines,

$$\frac{a}{\sin x} = \frac{g}{\sin E}$$

$$\frac{c}{\sin 90} = \frac{g}{\sin A}$$

$$\frac{c}{a} = \frac{\sin E}{\sin A \sin x} \qquad \text{(K-10)}$$

$$\frac{c}{a} \simeq \frac{E}{A} \; (\sin x \simeq 1) \qquad \text{(K-11)}$$

but by equation (K-7)

$$\frac{D}{E} = \frac{B}{A}$$

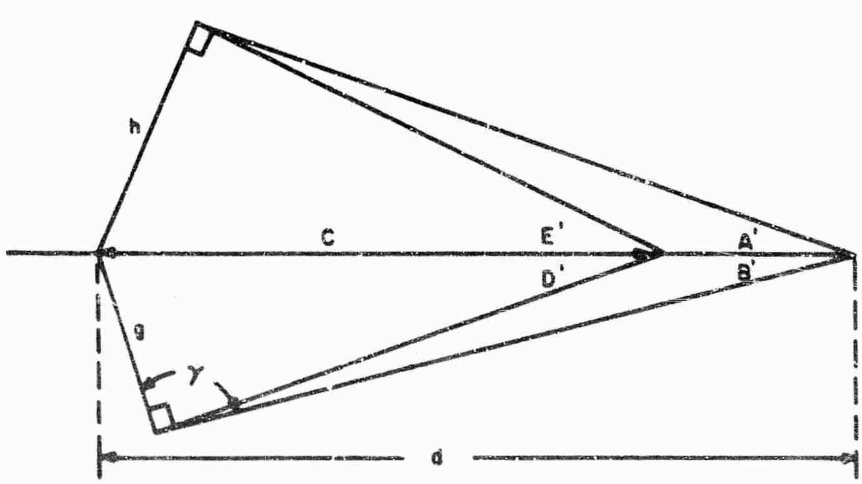
$$\frac{D + E}{E} = \frac{B + A}{A} \qquad \text{(K-12)}$$

$$\frac{E}{A} = \frac{D + E}{B + A} \qquad \text{(K-13)}$$

Therefore

$$c/a = \frac{D + E}{B + A} = \frac{\theta_Y}{\theta_X} \qquad \text{(K-14)}$$

Likewise in the other two triangles involving θ_Z and the other θ_X, we have



$$\frac{c}{\sin 90} = \frac{g}{\sin D'} \tag{K-15}$$

$$\frac{d}{\sin \gamma} = \frac{g}{\sin B'} \tag{K-16}$$

$$\frac{d}{c} = \frac{\sin D' \sin \gamma}{\sin B'} \simeq \frac{D'}{B'} \tag{K-17}$$

however, as before,

$$D' + E' = \theta_x \tag{K-18}$$

$$A' + B' = \theta_z \tag{K-19}$$

and

$$\frac{d}{c} = \frac{D'}{B'} \simeq \frac{\theta_x}{\theta_y} \tag{K-20}$$

Finally

$$\frac{d}{a} = \frac{c}{a} \times \frac{d}{c} = \frac{\theta_y}{\theta_x} \times \frac{\theta_x}{\theta_z}$$

$$\frac{d}{a} = \frac{\theta_y}{\theta_z} \tag{K-21}$$

To convert this to other system parameters consider

$$\theta_y + x + \mu + \phi' = 360 \left[\phi' = 180 - \theta_x\right] \tag{K-22}$$

or

$$\theta_y = 180 + \theta_x - (x + \mu)$$

and

$$\theta_z = x + \mu + \theta_x - 180 \tag{K-23}$$

Further

$$\frac{\Delta\theta_m}{2} = 90 - x \tag{K-24}$$

$$\frac{\Delta\theta_n}{2} = 90 - \mu \tag{K-25}$$

Giving

$$\frac{\Delta\theta_m + \Delta\theta_n}{2} = 180 - (x + \mu) \qquad \text{(K-26)}$$

$$-\left(\frac{\Delta\theta_m + \Delta\theta_n}{2}\right) = (x + \mu) - 180 \qquad \text{(K-27)}$$

Equations (K-14), (K-20) and (K-21) can be rewritten

$$\frac{c}{a} = \frac{\theta_y}{\theta_x} = \frac{\theta_x + \frac{\Delta\theta_m}{2} + \frac{\Delta\theta_n}{2}}{\theta_x} \qquad \text{(K-28)}$$

$$\frac{d}{a} = \frac{\theta_y}{\theta_x} = \frac{\theta_x}{\theta_x - \frac{\Delta\theta_m}{2} - \frac{\Delta\theta_n}{2}} \qquad \text{(K-29)}$$

and

$$\frac{d}{a} = \frac{\theta_x + \frac{\Delta\theta_m}{2} + \frac{\Delta\theta_n}{2}}{\theta_x - \frac{\Delta\theta_m}{2} - \frac{\Delta\theta_n}{2}} \qquad \text{(K-30)}$$

Finally, to a very close approximation,

$$\frac{\Delta\theta_m}{2} = \sin^{-1}\frac{m}{r_m} = \frac{|m|}{r_m}$$

$$\frac{\Delta\theta_n}{2} = \frac{|n|}{r_n}$$

and (with θ_x given in radians)

$$\frac{d}{a} = \frac{\theta_x + \frac{|n|}{r_n} + \frac{|m|}{r_m}}{\theta_x - \frac{|n|}{r_n} - \frac{|m|}{r_m}} \qquad \text{(K-31)}$$

and so forth.

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13. ABSTRACT

An engineering model is developed of the error contours encountered in position fixing using synchronous time of arrival data. Normal distribution is emphasized. Probabilities of fixes occurring within circles and ellipses are determined using the natural oblique coordinates associated with the measuring system. Comparisons are made of three-observer and four-observer configurations. Examples are provided and applications are discussed.

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